Mixing at 1-loop in a SU(2)_L gauge theory of weak interactions

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June 15th 2010

MIXING AT 1-LOOP IN A $SU(2)_L$ GAUGE THEORY OF WEAK INTERACTIONS

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Abstract: Flavor mixing is scrutinized at 1-loop in a $SU(2)_L$ gauge theory of massive fermions. The main issue is to cope with kinetic-like, momentum (p^2) dependent effective interactions that arise at this order. They spoil the unitarity of the connection between flavor and mass states, which potentially alters the standard Cabibbo-Kobayashi-Maskawa (CKM) phenomenology by giving rise, in particular, to extra flavor changing neutral currents (FCNC). We explore the conservative requirement that these should be suppressed, which yields relations between the CKM angles, the fermion and W masses, and a renormalization scale μ. For two generations, two solutions arise: either the mixing angle of the fermion pair the closer to degeneracy is close to maximal while, inversely, the mass and flavor states of the other pair are quasi-aligned, or mixing angles in both sectors are very small. For three generations, all mixing angles of neutrinos are predicted to be large ($|\theta_{23}| \approx$ maximal is the largest) and the smallness of their mass differences induces mass-flavor quasi-alignment for all charged leptons. The hadronic sector differs in that the top quark is twice as heavy as the W. The situation is, there, bleaker, as all angles come out too large, but, nevertheless, encouraging, because θ_{12} decreases as the top mass increases. Whether other super-heavy fermions could drag it down to realistic values stays an open issue, together with the role of higher order corrections. The same type of counterterms that turned off the 4th order static corrections to the quark electric dipole moment are, here too, needed, in particular to stabilize quantum corrections to mixing angles.

PACS: 12.15.Ff 12.15.Lk 14.60.Pq **Keywords:** mixing, radiative corrections, mass-splitting

1 Introduction

The origin of large mixing angles observed in leptonic charged currents is still largely unknown [1]. A widespread belief is that it is linked to a quasi-degeneracy of neutrinos, but this connection was never firmly established. And it cannot be on simple grounds. Indeed, the mixing angles that are "observed" in neutrino oscillations are the ones occurring in charged currents, which combine the individual mixing matrices of fermions with different electric charges [1] ; the path that goes from the quasi-degeneracy of one of the two doublets to large mixing in the PMNS matrix [2] cannot thus be completely straightforward. Furthermore, homographic transformations on a (mass) matrix, while changing its eigenvalues, do not change its eigenvectors, neither, accordingly, mixing angles; an infinity of different mass spectra can thus be associated with a given mixing angle.

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[1]The electronic (ν_e), muonic (ν_μ), and tau (ν_τ) neutrinos are defined as the neutrinos that couple, inside charged currents, to the mass eigenstates of charged leptons. They are accordingly related to the neutrino mass eigenstates by $(\nu_e\ \nu_\mu\ \nu_\tau)^T = K_\ell^\dagger K_\nu (\nu_{em}\ \nu_{\mu m}\ \nu_{\tau m})^T$ where K_ℓ and K_ν are the mixing matrices respectively of charged leptons and neutrinos. This connection is seen to involve the hermitian conjugate $K_\ell^\dagger K_\nu$ of the PMNS matrix.

We shall first focus on two pairs of fermions, making up two generations. For the sake of convenience (mainly for the simplicity of notations) we shall often call them generically (d, s) and (u, c). The first will be supposed to be close to degeneracy and the second largely split. Results are transposed to the leptonic sector: the Cabibbo angle θ_c [3] is then, in particular, replaced by the corresponding entry θ_{PMNS} of the (2×2) PMNS matrix. Results which are specific to neutrinos will of course be written with the adequate notations.

This study, which finally supports a relation between quasi-degeneracy and large mixing, rests on the following argumentation.

The physical states are the eigenstates of the propagator at its poles; in case of a coupled system of n particles, like massive fermions in the standard model of electroweak interactions [4] which are coupled through the scalar sector, the propagator, which is also the inverse of the quadratic Lagrangian, is a $n \times n$ matrix;
The determination of an orthogonal set of physical states accordingly requires the diagonalization of the sum of the kinetic terms and of the mass terms in the Lagrangian;
At the classical level, this procedure yields the standard Cabibbo-Kobayashi-Maskawa (CKM) [3] [5] phenomenology. The classical Lagrangian is written from the start devoid *a priori*, in bare flavor space, of FCNC. In direct connection with the unitarity of mixing matrices, in particular the Cabibbo matrix, the $SU(2)$ gauge algebra closes on a diagonal $\mathbb{T}^3$ generator, which eliminates FCNC at this order, in bare mass space as well as in bare flavor space [2]. FCNC are generated at 1-loop among bare flavor or mass states (see Fig. 1), but they are damped by the so-called "Cabibbo suppression". This phenomenology is, up to now, in very good agreement with experiment, and we choose to preserve it;
Subtle issues arise when considering the quadratic effective Lagrangian at 1-loop since, in particular, non-diagonal kinetic-like transitions are generated (Fig, 2). Then, the mandatory re-diagonalization of kinetic terms, which is generally overlooked, exhibits two main features. First, due to the presence of mass-splittings, it unavoidably involves slightly non-unitary transformations, which introduces in bare flavor space at 1-loop, a new set of, mass and mixing (and p^2) dependent, FCNC. Secondly, the 1-loop corrections to the mixing angles are non-perturbative and present a high instability in the vicinity of degeneracy. This strongly motivates the introduction of counterterms "à la Shabalin" [6] that cancel 1-loop non-diagonal transitions "on mass-shell".
They restore a quasi-standard Cabibbo phenomenology, but for the persistence of extra, mass and mixing dependent, FCNC in bare flavor space. Their occurring is rooted in the non-degeneracy of fermions, which counterterms cannot turn off. They are built to cancel non-diagonal 1-loop transitions when one of the two concerned external fermions is on mass-shell, but the second can, then, only be off mass-shell. So, while 1-loop mass eigenstates, which result from the diagonalization of the effective 1-loop Lagrangian, are, by definition, orthogonal and, as we show, do not exhibit FCNC [3], this is not exactly so for bare mass states: orthogonality only truly occurs among one on mass-shell and one off-mass shell fermion.
We investigate at which condition these extra FCNC can get suppressed. Such a requirement establishes a connection between mass splittings and the Cabibbo angle θ_c, which, for two generations and $m_u^2, m_d^2, m_s^2, m_c^2, p^2 \ll M_W^2$, writes $\cos 2\theta_c \approx -\frac{1}{2}\frac{m_s^2 - m_d^2}{m_c^2 - m_u^2}$. θ_c is seen to be quasi-maximal as soon as $|m_s - m_d| \ll |m_c - m_u|$, that is, when one of the two fermion pair is much closer to degeneracy than the second. A similar condition is realized in the 2-generation leptonic sector, pushing to large values the similar angle of the PMNS matrix. Thus, the conservative requirement that the standard classical Cabibbo phenomenology should be preserved at 1-loop provides, through FCNC, a connection between large mixing and the quasi-degeneracy of two same-charge fermions.

Nature is however more complex: – first, there are three and not only two generations; secondly, in the quark sector, all mixing angles are small; – last, while, in the lepton sector, the "atmospheric" angle θ_{23} seems actually close to maximal, this is not the case for the "solar" angle θ_{12} which, though large, looks

[2]The terminology FCNC is certainly not very good when dealing with (bare) mass states. The reader should understand it as "non-diagonal currents in mass space".

[3]with a subtlety, due to the dependence of p^2, that is evoked in appendix A.1.

closer to 35^o, nor for θ_{13}, which could be much smaller [7]. This is why the last part of this work is dedicated to the 3-generations case, making in particular the distinction between the leptonic case, where all known fermions stand well below the electroweak scale M_W, and the quark case where the top quark weights roughly $2M_W$.

This work is structured as follows. Sections 2 to 6 deal with two generations of fermions, first, from section 2 to 4, without introducing Shabalin's counterterms, then, in sections 5 and 6, in their presence. Section 7 analyzes in detail the case of three generations.

In section 2, we explain the procedure to re-diagonalize, at 1-loop, the quadratic Lagrangian (kinetic + mass terms) of an $SU(2)_L$ gauge theory for several generations of massive fermions. In subsection 2.1 we first briefly recall the standard procedure to diagonalize, by a bi-unitary transformation, the classical quadratic Lagrangian. We then outline, taking the example of two generations, how it is modified when 1-loop transitions introduce non-diagonal, p^2-dependent, kinetic-like interactions. In subsection 2.2 we give the analytical formulæ in the limit $p^2 \ll m_W^2$, which then largely simplify when the four fermions masses are much smaller than the W mass, too. Subsections 2.3 and 2.4 are respectively devoted to the re-diagonalization of kinetic terms, and of mass terms. The first are shown to unavoidably introduce, because of mass splittings, non-unitary transformations After these operations are done, the whole effective quadratic Lagrangian at 1-loop is back to diagonal, with its kinetic terms proportional; to the unit matrix $\mathbb{I}$.

In section 3, we focus on the (realistic) case $|m_s - m_d| \ll |m_c - m_u|$. We study individual mixing matrices (*i.e.* the ones in the (u,c) and (d,s) sectors) and the two corresponding mixing angles.

Section 4 is devoted to the 1-loop Cabibbo matrix. First we show how gauge invariance dictates the form of the 1-loop effective Lagrangian, by, in particular, relating through the covariant derivative, kinetic terms to gauge currents. We then demonstrate that, unlike individual mixing matrices, the Cabibbo matrix stays unitary at 1-loop.

In section 5, we first show that, in the absence of counterterms, the 1-loop renormalization of the mixing angle for degenerate (d,s) is pathological. We then show how the introduction of Shabalin's counterterms restore the stability and reliability of 1-loop corrections to mixing angles, in particular in the vicinity of degeneracy. The 1-loop Cabibbo matrix still keeps unitary in their presence.

In section 6, still for two generations, we show how extra FCNC arise, and we we solve the constraints controlling their suppression, first in the absence of counterterms, then in their presence.

Section 7 is an extensive study of the 3-generation case, in the presence of Shabalin's counterterms. In subsection 7.1, we write the three equations which guarantee that no extra FCNC is present in the bare flavor (or mass) space. We then explicitly list all possible solutions. In subsection 7.2 we give analytical expressions concerning 1-loop transitions between fermions when one among the six fermions making up three generations (the top quark) is heavier than the W. In subsection 7.3 we solve the constraints for quarks. In subsection 7.4 we solve them for neutrinos.

The conclusions and outlook are given in section 8. We also give, there, a comparison between this work and previous approaches concerning the renormalization of mixing angles.

In appendix A, we briefly comment on the dependence on p^2 and some of its consequences, that we neglected in the core of the paper where we considered the limit $p^2 \ll m_W^2$.

For the sake of simplicity (like in [6]), we work in a pure $SU(2)_L$ theory of weak interactions instead of the standard $SU(2)_L \times U(1)$ electroweak model [4]. Since the theory is renormalizable, we use the unitary gauge, devoid of the intricacies due to scalar fields and which, consistently working at order g^2, yields finite results for the quantities of concern to us. While we cannot, accordingly, verify the gauge independence of the results (independence on the ξ parameter in an R_ξ gauge), gauge invariance is of primordial importance.

2 1-loop transitions between non-degenerate fermions ; re-diagonalizing the quadratic Lagrangian

2.1 Principle of the method

At the classical level, a bi-unitary transformation is used, in flavor space, to diagonalize the sum of kinetic + mass terms $\begin{pmatrix} \bar d_f^0 & \bar s_f^0 \end{pmatrix} \left[\not p\, \mathbb{I} - M_f^0\right] \begin{pmatrix} d_f^0 \\ s_f^0 \end{pmatrix}$ into $\begin{pmatrix} \bar d_m^0 & \bar s_m^0 \end{pmatrix} \left[\not p\, \mathbb{I} - \begin{pmatrix} m_d & \\ & m_s \end{pmatrix}\right] \begin{pmatrix} d_m^0 \\ s_m^0 \end{pmatrix}$. The two unitary transformations, acting respectively on right- and left-handed fermions, preserve the canonical form of both kinetic terms, which stay proportional to the unit matrix $\mathbb{I}$. This defines the classical masses m_d and m_s. The corresponding classical mass eigenstates d_m^0 and s_m^0 are orthogonal with respect to the classical Lagrangian, which is akin to the property that no transition between them occurs at the classical level. In particular, the classical Lagrangian in flavor space is written devoid *a priori* of FCNC; this is directly related to the property that kinetic terms are proportional to the unit matrix, since gauge currents are simply deduced by introducing the covariant derivative with respect to the gauge group. The above diagonalization leads to the standard Cabibbo (or CKM) phenomenology, in which, in particular, non-diagonal neutral gauge currents only get generated at 1-loop (see Fig. 1), and are damped, when expressed in bare mass space, by the so-called "Cabibbo suppression". This phenomenology is, up to now, in agreement with experiment.

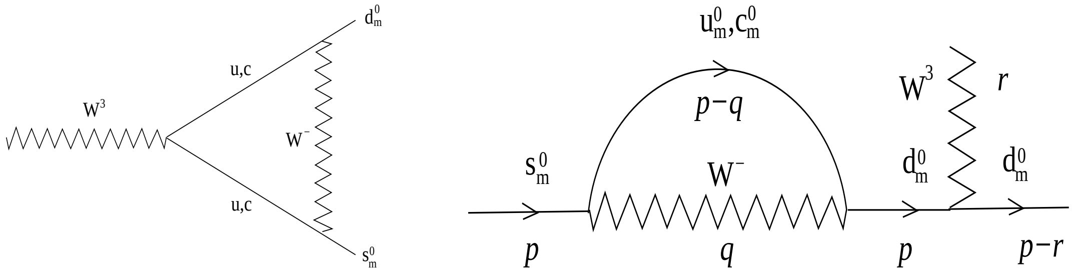

Fig. 1: "Standard" flavor changing neutral currents at 1-loop

However, 1-loop non-diagonal transitions, like $s_m^0 \to d_m^0$ depicted in Fig. 2, trigger new phenomena which have not yet been fully considered and which, in particular, also generate FCNC. By the effect of the corresponding renormalization, the kinetic terms of left-handed fermions stay indeed no longer proportional to the unit matrix $\mathbb{I}$ but to some non-diagonal $K_d = \mathbb{I} + H_d, H_d = \mathcal{O}(g^2)$, which depends on the classical masses (fermions and gauge fields), on the classical Cabibbo mixing angle θ_c, and on p^2. The pure kinetic terms K_d for (d_m^0, s_m^0) written in (6) [4] can be cast back to their canonical form by a p^2-dependent non-unitary transformation $\mathcal{V}_d(p^2, \ldots)$ according to

$$\mathcal{V}_d^\dagger\, K_d\, \mathcal{V}_d = \mathbb{I}. \tag{1}$$

By (1), which entails $K_d = (\mathcal{V}_d^{-1})^\dagger \mathcal{V}_d^{-1}$, the kinetic terms [5] $(\overline{d_{mL}^0}, \overline{s_{mL}^0}) K_d \not p \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix}$ at 1-loop for left-handed d and s in the bare mass basis rewrite $(\overline{d_{mL}^0}, \overline{s_{mL}^0})(\mathcal{V}_d^{-1})^\dagger \mathcal{V}_d^{-1} \not p \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix}$, which leads to

[4] For the sake of convenience, we work in the bare mass basis.

[5] The subscript "$_L$" refers to left-handed fermions and "$_R$" to right-handed ones.

defining d^1_{mL} and s^1_{mL} such that $\left(\begin{array}{c} d^1_{mL} \\ s^1_{mL}\end{array}\right) = \mathcal{V}_d^{-1}\left(\begin{array}{c} d^0_{mL} \\ s^0_{mL}\end{array}\right)$. The mass matrix, which had been made diagonal in the classical basis (d^0_m, s^0_m), is no longer so in the basis (d^1_{mL}, s^1_{mL}). The second step of the procedure is accordingly to re-diagonalize it by a second bi-unitary transformation. It leaves unchanged the canonical form of the kinetic terms that has been rebuilt in the first step of the procedure. After the two steps have been completed, the sum of kinetic + mass terms at 1-loop is diagonal. The resulting basis of 1-loop mass eigenstates $(d_{mL}(p^2,\ldots), s_{mL}(p^2,\ldots))$ is such that, at this order and at any given p^2, there exists no transition between d_{mL} and s_{mL}. They are thus, by definition, orthogonal at 1-loop.

2.2 1-loop transitions: explicit calculations

We now explicitly calculate 1-loop transitions. Gauge interactions induce diagonal and non-diagonal transitions between bare mass states. For example, Fig. 2 describes non-diagonal $s^0_m \to d^0_m$ transitions, mediated by the $W^\pm$ gauge bosons. Diagonal transitions are mediated either by $W^\pm_\mu$ or by W^3_μ.

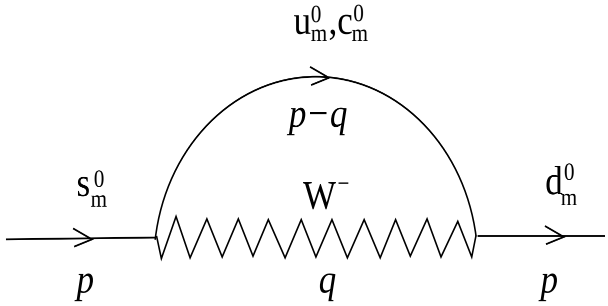

Fig. 2: $s^0_m \to d^0_m$ transition at 1-loop

The one depicted in Fig. 2 contributes as a left-handed, kinetic-like, p^2-dependent interaction

$$\mathcal{A}_{sd}\, \bar{d}^0_m \not{p}(1-\gamma_5)\, s^0_m, \quad \mathcal{A}_{sd} = \sin\theta_c \cos\theta_c \big(h(p^2, m_u, m_W) - h(p^2, m_c, m_W)\big), \tag{2}$$

that we abbreviate, with shortened notations $\sin\theta_c \equiv s_c, \cos\theta_c \equiv c_c$, into

$$\mathcal{A}_{sd} = s_c c_c (h_u - h_c). \tag{3}$$

It depends in particular on the classical Cabibbo angle $\theta_c = \theta_d - \theta_u$. The function h is dimensionless.

It is straightforward to deduce that all (diagonal and non-diagonal) 1-loop transitions between s^0_m and d^0_m mediated by $W^\pm$ gauge bosons transform their kinetic terms into

$$\begin{aligned} &\left(\begin{array}{cc} \bar{d}^0_m & \bar{s}^0_m \end{array}\right)\left[\mathbb{I}\not{p} + \left(\begin{array}{cc} c_c^2 h_u + s_c^2 h_c & s_c c_c (h_u - h_c) \\ s_c c_c (h_u - h_c) & s_c^2 h_u + c_c^2 h_c \end{array}\right)\not{p}(1-\gamma_5)\right]\left(\begin{array}{c} d^0_m \\ s^0_m \end{array}\right) \\ = \;& \left(\begin{array}{cc} \bar{d}^0_m & \bar{s}^0_m \end{array}\right)\left[\mathbb{I}\not{p} + \left(\frac{h_u + h_c}{2} + (h_u - h_c)\, \mathcal{T}_x(2\theta_c)\right)\not{p}(1-\gamma_5)\right]\left(\begin{array}{c} d^0_m \\ s^0_m \end{array}\right), \end{aligned} \tag{4}$$

where we noted

$$\mathcal{T}_x(\varphi) = \frac{1}{2}\left(\begin{array}{cc} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{array}\right). \tag{5}$$

To the contributions (4) we must add the diagonal transitions mediated by the W^3_μ gauge boson. The kinetic terms for left-handed d^0_m and s^0_m quarks then become (omitting the fermionic fields and the dependence on $p^2,\ldots$) [6]

$$
\begin{aligned}
K_d &= \mathbb{I} + H_d \; ;\\
H_d &= \begin{pmatrix} \mathcal{A}_{dd} & \mathcal{A}_{ds} \\ \mathcal{A}_{sd} & \mathcal{A}_{ss} \end{pmatrix} = \frac{h_u + h_c}{2} + (h_u - h_c)\, \mathcal{T}_x(2\theta_c) + \frac{1}{2}\begin{pmatrix} h_d & \\ & h_s \end{pmatrix},
\end{aligned}
\tag{6}
$$

where $h_d = h(p^2, m_d, m_W)$ and $h_s = h(p^2, m_s, m_W)$. Likewise, in the (u,c) sector, one has

$$
\begin{aligned}
K_u &= \mathbb{I} + H_u \; ;\\
H_u &= \begin{pmatrix} \mathcal{A}_{uu} & \mathcal{A}_{uc} \\ \mathcal{A}_{cu} & \mathcal{A}_{cc} \end{pmatrix} = \frac{h_d + h_s}{2} + (h_d - h_s)\, \mathcal{T}_x(-2\theta_c) + \frac{1}{2}\begin{pmatrix} h_u & \\ & h_c \end{pmatrix}.
\end{aligned}
\tag{7}
$$

Explicitly, one has

$$
\begin{aligned}
\mathcal{A}_{sd} \quad &= \quad \frac{g^2}{4}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_W^2}\Big[(2-\epsilon)(\not p - \not q) + \frac{2q.(p-q)\not p - q^2(\not p - \not q)}{m_W^2}\Big](1-\gamma^5)\\
&\qquad\qquad \Big[\frac{V_{us}V^*_{ud}}{(p-q)^2 - m_u^2} + \frac{V_{cs}V^*_{cd}}{(p-q)^2 - m_c^2}\Big]\\
&\overset{unitarity\ of\ V}{=} \frac{g^2}{4}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_W^2}\Big[(2-\epsilon)(\not p - \not q) + \frac{2q.(p-q)\not p - q^2(\not p - \not q)}{m_W^2}\Big](1-\gamma^5)\\
&\qquad\qquad V_{us}V^*_{ud}\frac{m_u^2 - m_c^2}{\big[(p-q)^2 - m_u^2\big]\big[(p-q)^2 - m_c^2\big]}.
\end{aligned}
\tag{8}
$$

The factor $V_{us}V^*_{ud}$ in (8) is the $s_c c_c$ of (2), which finally defines $(h_u - h_c)$ of (3).
All our forthcoming results depend on differences like $(h_i - h_j)$. In the unitary gauge, after introducing 2 Feynman parameters x and y, the dimensionally (for $n = 4 - \epsilon$ dimension) regularized expression for $(h_i - h_j)$ writes ($\gamma \approx 0.572$ is the Euler constant)

$$
\begin{aligned}
h_i - h_j \;&=\; \frac{g^2}{4}\frac{i}{16\pi^2}(m_i^2 - m_j^2)\int_0^1 dx \int_0^1 dy\ 2y\\
&\Bigg[-(1-y)\left(1 + \frac{y^2p^2}{2m_W^2}\right)\frac{1}{R^2} + \frac{1}{m_W^2}\Bigg(-\left(-\frac{1+3y}{2}\right)\left(\frac{2}{\epsilon} + \ln 4\pi - \gamma\right) + \frac{1+y}{2} + \frac{1+3y}{2}\ln\frac{R^2}{\mu^2}\Bigg)\Bigg],\\
R^2 \;&=\; -y(1-y)p^2 + y(1-x)m_j^2 + xym_i^2 + (1-y)m_W^2.
\end{aligned}
\tag{9}
$$

To obtain (9), the relation $\gamma_\nu\gamma_\alpha\gamma_\nu = -(2-\epsilon)\gamma_\alpha$ between the Dirac matrices has been used. The scale μ originates from the necessity, in $4-\epsilon$ dimensions, to replace g^2 by $g^2\mu^\epsilon$. The exact analytical expression for all values of p^2, m_i^2, m_j^2 cannot be easily obtained, but, when $p^2 \ll m_W^2$, $y(1-y)p^2$ can be safely neglected with respect to $(1-y)m_W^2$ in R^2, such that (9) simplifies into (we write this time its expression once renormalized in the $\overline{MS}$ scheme which amounts to eliminating from (9) the pole in $1/\epsilon$ together

[6] From now onwards, to lighten the notations, we shall frequently omit the dependence on p^2 and on the masses.

with the terms proportional to $\ln 4\pi - \gamma$)

$$h_i - h_j \overset{p^2 \ll m_W^2}{\underset{\overline{MS}}{\approx}} \frac{g^2}{4}\frac{i}{16\pi^2}(m_i^2 - m_j^2)\int_0^1 dx \int_0^1 dy\; 2y\left[-\frac{1-y}{r^2} + \frac{1}{2m_W^2}\left((1+y) + (1+3y)\ln\frac{r^2}{\mu^2}\right)\right],$$
$$r^2 = y(1-x)m_j^2 + xym_i^2 + (1-y)m_W^2. \tag{10}$$

The integration over x can be done explicitly. This leads to the expression

$$h_i - h_j \overset{p^2 \ll m_W^2}{\underset{\overline{MS}}{\approx}} \frac{g^2}{4}\frac{i}{16\pi^2}\int_0^1 dy\left[-2(1-y)\ln\frac{ym_i^2 + (1-y)m_W^2}{ym_j^2 + (1-y)m_W^2} - 2y^2\frac{m_i^2 - m_j^2}{m_W^2}\right.$$
$$\left. y(1+3y)\left(\frac{m_i^2}{m_W^2}\ln\frac{ym_i^2 + (1-y)m_W^2}{\mu^2} - \frac{m_j^2}{m_W^2}\ln\frac{ym_j^2 + (1-y)m_W^2}{\mu^2}\right)\right], \tag{11}$$

which, like (9) and (10), vanishes when $m_i = m_j$. After explicitly doing the $\int dy$ integration, one gets

$$h_i - h_j \overset{p^2 \ll m_W^2}{\underset{\overline{MS}}{\approx}} \frac{g^2}{4}\frac{i}{16\pi^2}\left[-\frac{2}{3}\frac{m_i^2 - m_j^2}{m_W^2} - 2\left(\frac{m_W^2 \ln\frac{m_W^2}{\mu^2} - m_i^2\ln\frac{m_i^2}{\mu^2}}{m_W^2 - m_i^2} - (i \leftrightarrow j)\right)\right.$$
$$+\left(\left(2 + \frac{m_i^2}{m_W^2}\right)\left(-\frac{m_W^2}{m_W^2 - m_i^2} + \frac{m_W^2\left(m_W^2\ln\frac{m_W^2}{\mu^2} - m_i^2\ln\frac{m_i^2}{\mu^2}\right)}{(m_W^2 - m_i^2)^2}\right.\right.$$
$$\left.\left. + \frac{1}{4}\frac{m_W^2 + m_i^2}{m_W^2 - m_i^2} - \frac{1}{2}\frac{m_W^4\ln\frac{m_W^2}{\mu^2} - m_i^4\ln\frac{m_i^2}{\mu^2}}{(m_W^2 - m_i^2)^2}\right) - (i \leftrightarrow j)\right)$$
$$+\left(\frac{m_i^2}{m_W^2}\frac{1}{(m_W^2 - m_i^2)^2}\left(-\frac{11m_W^4 - 7m_W^2 m_i^2 + 2m_i^4}{6}\right.\right.$$
$$\left.\left.\left. + \frac{m_W^6\ln\frac{m_W^2}{\mu^2} + \left(-3m_W^4 m_i^2 + 3m_W^2 m_i^4 - m_i^6\right)\ln\frac{m_i^2}{\mu^2}}{m_W^2 - m_i^2}\right) - (i \leftrightarrow j)\right)\right]. \tag{12}$$

Eq. (12) is only valid for $p^2 \ll m_W^2$ but its dependence on the fermion masses m_i and m_j is then exact. In the limit, always valid for 2 generations, when $m_i^2, m_j^2 \ll m_W^2$, it drastically simplifies to

$$h_i - h_j \overset{p^2, m_i^2, m_j^2 \ll m_W^2}{\underset{\overline{MS}}{\approx}} \frac{g^2}{4}\frac{i}{16\pi^2}\frac{m_i^2 - m_j^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right). \tag{13}$$

In the case of 3 generations of quarks, the top quark enters the game and one is in the situation when $p^2, m_i^2 \ll m_W^2$ but $m_j^2 \equiv m_t^2 \geq m_W^2$. The corresponding formulæ will be given in subsection 7.2. Note that, in the approximation $p^2 \ll m_W^2$ that we are using, the final result (13) no longer depends on p^2.

2.3 First step: re-diagonalizing kinetic terms back to the unit matrix

We shall now diagonalize the quadratic part of the effective 1-loop Lagrangian, which means putting the pure kinetic terms back to the unit matrix and, at the same time, re-diagonalizing the mass matrix. This is accordingly a two-steps procedure.

Since the kinetic terms of right-handed fermions are not modified, we shall only be concerned with the left-handed ones.

The pure kinetic terms K_d for (d_m^0, s_m^0) written in (6) can be cast back to their canonical form by a p^2-dependent non-unitary transformations $\mathcal{V}_d(p^2, \ldots)$ according to (1).

The procedure to find $\mathcal{V}_d$ is the following. Let $(1+t^d_+)$ and $(1+t^d_-)$, $t^d_+, t^d_- = \mathcal{O}(g^2)$, be the eigenvalues of the symmetric matrix K_d; explicitly

$$t^d_\pm = \frac{h_u + h_c + \left[\frac{h_d+h_s}{2}\right]}{2} \pm \frac{1}{2}\sqrt{(h_u-h_c)^2 + \left[\frac{h_d-h_s}{2}\right]^2 + 2\,(h_u-h_c)\,\left[\frac{h_d-h_s}{2}\right]\cos 2\theta_c}. \quad (14)$$

K_d can be diagonalized by a rotation $\mathcal{R}(\omega_d) \equiv \begin{pmatrix} \cos\omega_d & \sin\omega_d \\ -\sin\omega_d & \cos\omega_d \end{pmatrix}$ according to

$$\mathcal{R}(\omega_d)^\dagger\, K_d\, \mathcal{R}(\omega_d) = \begin{pmatrix} 1+t^d_+ & \\ & 1+t^d_- \end{pmatrix}, \quad (15)$$

with

$$\tan 2\omega_d = \frac{-(h_u-h_c)\sin 2\theta_c}{(h_u-h_c)\cos 2\theta_c + \left[\frac{h_d-h_s}{2}\right]}, \quad (16)$$

or, equivalently,

$$\cos 2\omega_d = \frac{(h_u-h_c)\cos 2\theta_c + \left[\frac{h_d-h_s}{2}\right]}{t^d_+ - t^d_-}, \quad \sin 2\omega_d = -\frac{(h_u-h_c)\sin 2\theta_c}{t^d_+ - t^d_-}, \quad (17)$$

in which $(t^d_+ - t^d_-)$ can be immediately obtained from (14) [7] .
Eq. (16) defines ω_d in particular as a function of θ_c, $\omega_d = \omega_d(\theta_c, \ldots)$. Since both numerator and denominator of (16) are $\mathcal{O}(g^2)$, ω_d does not depend on the coupling constant g.
The diagonal matrix obtained in (15) is not yet the required unit matrix, but one simply gets to it by renormalizing the columns of $\mathcal{R}(\omega_d)$ respectively by $\frac{1}{\sqrt{1+t^d_+}}$ and $\frac{1}{\sqrt{1+t^d_-}}$. The looked-for non-unitary matrix $\mathcal{V}_d$ writes finally

$$\mathcal{V}_d = \begin{pmatrix} \dfrac{c_{\omega_d}}{\sqrt{1+t^d_+}} & \dfrac{s_{\omega_d}}{\sqrt{1+t^d_-}} \\ -\dfrac{s_{\omega_d}}{\sqrt{1+t^d_+}} & \dfrac{c_{\omega_d}}{\sqrt{1+t^d_-}} \end{pmatrix}. \quad (18)$$

It differs from the rotation $\mathcal{R}(\omega_d)$ only at $\mathcal{O}(g^2)$ and satisfies

$$\mathcal{V}_d\,\mathcal{V}_d^\dagger = \frac{1}{(1+t^d_+)(1+t^d_-)}\left(\mathbb{I} + \frac{t^d_+ + t^d_-}{2} - (t^d_+ - t^d_-)\;\mathcal{T}_x(-2\omega_d)\right), \quad \mathcal{V}_d^\dagger\,\mathcal{V}_d = \begin{pmatrix} \dfrac{1}{1+t^d_+} & \\ & \dfrac{1}{1+t^d_-} \end{pmatrix}. \quad (19)$$

For $|m_d^2 - m_s^2| \ll |m_u^2 - m_c^2|$, $|h_d - h_s| \ll |h_u - h_c|$, $(t_+ - t_-) \approx (h_u - h_c)$ and the expression for $\sin 2\omega_d$ in (17) shows that $\omega_d(\theta_c) \approx -\theta_c$. So, when the pair (d,s) is close to degeneracy and (u,c) far from it, $\mathcal{V}_d$ becomes close to a rotation $\mathcal{R}(-\theta_c)$. We shall come back on this case in subsection 5.1.
Eq. (19) shows that mass splittings $(t_+ \neq t_-)$ are responsible for the non-unitarity of $\mathcal{V}$, and, so, for the non-unitary relation between 1-loop and bare mass states (the same occurs in flavor space). Note that this non-unitarity persists when $\omega_d \to 0$, which will be the case when counterterms are introduced (see subsection 6.2). Unitarity can only be achieved for $t_+ = t_-$; according to (14), this requires $(h_u-h_c)^2 + \left[\frac{h_d-h_s}{2}\right]^2 + 2(h_u-h_c)\left[\frac{h_d-h_s}{2}\right]\cos 2\theta_c = 0$, which, since $\cos 2\theta_c \in [-1, +1]$, can only eventually occur:
– either for $(h_u - h_c) = \frac{h_d-h_s}{2}$, that is, for $(m_u - m_c) = \frac{m_d - m_s}{\sqrt{2}}$, in which case $\cos 2\theta_c = -1 \Leftrightarrow \theta_c = \pi$;
– or for $h_u = h_c, h_d = h_s \Leftrightarrow m_u = m_c, m_d = m_s$ (twice degenerate system).

[7] Eq. (16) also rewrites $\frac{\sin 2(\omega_d+\theta_c)}{\sin 2\omega_d} = -\frac{h_d-h_s}{h_u-h_c}$, which shows that $\omega_d \to -\theta_c$ when $|m_s - m_d| \ll |m_u - m_c|$.

2.4 Second step: re-diagonalizing the mass matrix

2.4.1 1-loop mass eigenstates

As mentioned in subsection 2.1, the re-diagonalization of kinetic terms leads to defining the basis (d^1_{mL}, s^1_{mL}), which is related to the bare mass basis by the non-unitary relation $\mathcal{V}_d$. In this basis, the mass terms $(\overline{d^0_{mL}}, \overline{s^0_{mL}}) M_d \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} + h.c.$, with $M_d = diag(m_d, m_s)$, rewrite $(\overline{d^1_{mL}}, \overline{s^1_{mL}}) \mathcal{V}_d^\dagger M_d \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} + h.c.$. Hence, the mass matrix that needs to be re-diagonalized is $\mathcal{V}_d^\dagger M_d$. It is done through two unitary transformations $\mathcal{R}(\xi_d)$ and $S(\xi_d)$ such that $\mathcal{R}(\xi_d)^\dagger (\mathcal{V}_d^\dagger M_d) S(\xi_d) = diag(\mu_d, \mu_s)$. Since $\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d$ is a real symmetric matrix

$$\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d = \mathcal{V}_d^\dagger \begin{pmatrix} m_d^2 & \\ & m_s^2 \end{pmatrix} \mathcal{V}_d = \begin{pmatrix} \dfrac{m_d^2 c_{\omega_d}^2 + m_s^2 s_{\omega_d}^2}{1+t_+^d} & -\dfrac{s_{\omega_d} c_{\omega_d} (m_s^2 - m_d^2)}{\sqrt{(1+t_+^d)(1+t_-^d)}} \\ -\dfrac{s_{\omega_d} c_{\omega_d} (m_s^2 - m_d^2)}{\sqrt{(1+t_+^d)(1+t_-^d)}} & \dfrac{m_d^2 s_{\omega_d}^2 + m_s^2 c_{\omega_d}^2}{1+t_-^d} \end{pmatrix}, \tag{20}$$

$\mathcal{R}(\xi_d)$ can be taken as a rotation, according to

$$\mathcal{R}(\xi_d)^\dagger \left(\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d \right) \mathcal{R}(\xi_d) = \begin{pmatrix} \mu_d^2 & \\ & \mu_s^2 \end{pmatrix}. \tag{21}$$

Being unitary, it preserves the canonical form of the kinetic terms that had been rebuilt in subsection 2.3. It satisfies

$$\tan 2\xi_d = \frac{-(m_d^2 - m_s^2)\sqrt{(1+t_+^d)(1+t_-^d)}\sin 2\omega_d}{(m_d^2 - m_s^2)\left(1 + \dfrac{t_+^d + t_-^d}{2}\right)\cos 2\omega_d - (m_d^2 + m_s^2)\dfrac{t_+^d - t_-^d}{2}}. \tag{22}$$

Through $\omega_d(\theta_c, \ldots)$, (22) defines ξ_d in particular as a function of θ_c, $\xi_d = \xi_d(\theta_c, \ldots)$.

Since the mass terms rewrite $(\overline{d^1_{mL}}, \overline{s^1_{mL}}) \mathcal{R}(\xi_d)\ diag(\mu_d, \mu_s)\ S(\xi_d)^\dagger \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} + h.c.$, the 1-loop left-handed mass eigenstates (d_{mL}, s_{mL}) are defined by $(\overline{d_{mL}}, \overline{s_{mL}}) = (\overline{d^1_{mL}}, \overline{s^1_{mL}}) \mathcal{R}(\xi_d)$, which leads to

$$\begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} = \mathcal{V}_d \mathcal{R}(\xi_d) \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}. \tag{23}$$

By construction, at this order, there exists no transition between d_{mL} and s_{mL}, which are thus, by definition, orthogonal.

2.4.2 1-loop masses

The re-diagonalization of kinetic terms indirectly contributes to a renormalization of the masses: $m_d \to \mu_d, m_s \to \mu_s$. For $\frac{t_+^d - t_-^d}{2} \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} \cos 2\omega_d \ll 1$ and $\frac{t_+^d - t_-^d}{2} \frac{m_s^2 + m_d^2}{m_s^2 - m_d^2} \cos 2\omega_d \ll 1$ [8], one gets, when $m_d \neq$

[8] The first condition is immediately seen to be always satisfied. The second too, unless (d, s) are extremely close to degeneracy or degenerate, which does not occur for any known fermions.

m_s, from (20)

$$\begin{aligned}\mu_s^2 &\approx m_s^2\left(1-\frac{t_+^d+t_-^d}{2}\right)-m_d^2\,\frac{t_+^d-t_-^d}{2}\cos 2\omega_d,\\ \mu_d^2 &\approx m_d^2\left(1-\frac{t_+^d+t_-^d}{2}\right)+m_s^2\,\frac{t_+^d-t_-^d}{2}\cos 2\omega_d.\end{aligned} \tag{24}$$

This yields in particular, still when the two conditions mentioned at the beginning of this subsection are satisfied,

$$\frac{\mu_s^2-\mu_d^2}{\mu_s^2+\mu_d^2}\approx\frac{m_s^2-m_d^2}{m_s^2+m_d^2}-(t_+^d-t_-^d)\frac{m_s^4+m_d^4}{(m_s^2+m_d^2)^2}\cos 2\omega_d, \tag{25}$$

which becomes, for $m_s \approx m_d$ ($m_s \neq m_d$)

$$\begin{aligned}\frac{\mu_s^2-\mu_d^2}{\mu_s^2+\mu_d^2} &\stackrel{m_s\approx m_d}{\approx} \frac{m_s^2-m_d^2}{m_s^2+m_d^2}-\frac{t_+^d-t_-^d}{2}\cos 2\omega_d\\ &\stackrel{(17)}{\approx} \frac{m_s^2-m_d^2}{m_s^2+m_d^2}-\frac{1}{2}(h_u-h_c)\cos 2\theta_c=\frac{m_s^2-m_d^2}{m_s^2+m_d^2}+\frac{g^2}{16\pi^2}\frac{m_c^2-m_u^2}{m_W^2}\cos 2\theta_c.\end{aligned} \tag{26}$$

Supposing $\cos 2\theta_c > 0$ and $m_c > m_u$, $\frac{\mu_s^2-\mu_d^2}{\mu_s^2+\mu_d^2}$ goes to a minimum, identical to its classical value, when θ_c becomes maximal. A similar property is satisfied in the case of the MSW resonance (see for example [7]).

The classically degenerate case $m_d = m_s$ is most easily studied directly from (20). Degeneracy gets lifted at 1-loop since the renormalized masses become, then, $\mu_d^2 = \frac{m_{d,s}^2}{1+t_+^d}, \mu_s^2 = \frac{m_{d,s}^2}{1+t_-^d}$, such that $\frac{\mu_s^2-\mu_d^2}{\mu_s^2+\mu_d^2} \approx \frac{h_c-h_u}{2} \approx \frac{g^2}{16\pi^2}\frac{m_c^2-m_u^2}{m_W^2}$. It turns out to be the limit of (26) for $m_d = m_s$ and vanishing θ_c.

3 Individual mixing matrices and mixing angles at 1-loop

3.1 1-loop and classical mass eigenstates are non-unitarily related

According to (23), the left-handed 1-loop mass eigenstates (d_{mL}, s_{mL}) are related to the bare ones via the product of a non-unitary transformation $\mathcal{V}_d$ by a unitary one $\mathcal{R}(\xi_d)$. The two bases are accordingly non-unitarily related [8].

We recall (see subsection 2.3 after (19)) that mass splittings are at the origin of the non-unitarity of $\mathcal{V}_d$. [9] [10] [11].

Since bare mass states are related to bare flavor states by the classical mixing matrix $\mathcal{C}_{d0} \equiv \mathcal{R}(\theta_d)$ of the (d, s) pair, which is unitary, the physical mass eigenstates are also non-unitarily related to the latter. The relation is

$$\begin{pmatrix} d_{fL}^0 \\ s_{fL}^0 \end{pmatrix} = \mathcal{C}_{d0}\begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix} \stackrel{(23)}{=} \mathcal{C}_{d0}\,\mathcal{V}_d\,\mathcal{R}(\xi_d)\begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}, \tag{27}$$

3.2 Individual mixing matrices and mixing angles at 1-loop

3.2.1 The (d, s) mixing angle

According to (27), the individual mixing matrix at 1-loop is given by

$$\mathcal{C}_d = \mathcal{C}_{d0}\,\mathcal{V}_d\,\mathcal{R}(\xi_d) = \mathcal{R}(\theta_d)\,\mathcal{V}_d\,\mathcal{R}(\xi_d). \tag{28}$$

Since $\mathcal{V}_d \approx \mathcal{R}(\omega_d) + \mathcal{O}(g^2)$ (see (18)), $\mathcal{C}_d$, though slightly non-unitary, stays nevertheless close to a rotation

$$\mathcal{C}_d \approx \mathcal{R}(\theta_d + \omega_d + \xi_d) + \mathcal{O}(g^2). \tag{29}$$

The quantity $(\omega_d + \xi_d)$ is seen to renormalize the classical mixing angle θ_d; it satisfies, from (22), the relation (neglecting the terms proportional to $\frac{t_+ + t_-}{2}$ which are of order $g^{>2}$)

$$\tan 2(\omega_d + \xi_d) \approx \frac{-\tan 2\omega_d \left[\frac{t_+^d - t_-^d}{2}\ \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}\ \frac{1}{\cos 2\omega_d}\right]}{1 + \tan^2 2\omega_d - \left[\frac{t_+^d - t_-^d}{2}\ \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}\ \frac{1}{\cos 2\omega_d}\right]}. \tag{30}$$

In practice, $\tan 2(\omega_d + \xi_d)$ stays small, and so does, accordingly, $(\omega_d + \xi_d)$. Renormalization effects could become large only close to the pole of (30). It occurs for

$$\frac{1}{\cos 2\omega_d} = \frac{t_+^d - t_-^d}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}, \tag{31}$$

that is, for $\frac{1}{\cos 2\omega_d} = \mathcal{O}(g^2) \times \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}$, which is usually unphysical because it corresponds to $|\cos 2\omega_d| > 1$. $|\cos 2\omega_d|$ could become smaller than 1 only if, generically, $\left|\frac{m_d^2 - m_s^2}{m_d^2 + m_s^2}\right| < \frac{t_+^d - t_-^d}{2} \approx \frac{g^2}{16\pi^2} \frac{m_c^2 - m_u^2}{m_W^2}$, which is never satisfied for known fermions, quarks or leptons [9] .
From (30), (16) and (17) one also gets $\tan 2(\omega_d + \xi_d)$ as a function of θ_c and the classical masses

$$\tan 2(\omega_d + \xi_d) \approx \frac{\frac{1}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} (h_u - h_c) \sin 2\theta_c}{1 - \frac{1}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} \left((h_u - h_c) \cos 2\theta_c + \left[\frac{h_d - h_s}{2}\right]\right)}. \tag{32}$$

3.2.2 The (u, c) mixing angle

In the same configuration $|m_d - m_s| \ll |m_u - m_c|$, from the expression equivalent to (16) in the (u, c) sector, $\tan 2\omega_u = \frac{(h_d - h_s) \sin 2\theta_c}{(h_d - h_s) \cos 2\theta_c + \left[\frac{h_u - h_c}{2}\right]}$, one deduces that, since $|h_u - h_c| \gg |h_d - h_s|$, $\omega_u \to 0$. Then, from the equivalent of (32), one gets $\tan 2(\omega_u + \xi_u) \approx \frac{1}{2}(h_d - h_s) \sin 2\theta_c$, which is very small (see (13)).

4 The 1-loop Cabibbo matrix $\mathfrak{C}(p^2, \ldots)$

4.1 The effective Lagrangian at 1-loop (in the bare mass basis)

$SU(2)_L$ gauge invariance demands the replacement, in the Lagrangian, of the partial derivative ∂ by the covariant derivative D. This is how, at the classical level and in the bare mass basis, calling $\Psi_m^{0\,T} = (u_{mL}^0, c_{mL}^0, d_{mL}^0, s_{mL}^0)$, the kinetic + gauge terms write in their standard form
$i\,\overline{\Psi}_m^0 \overleftrightarrow{D_\mu} \gamma^\mu \Psi_m^0 \equiv \frac{i}{2}\Big(\overline{\Psi}_m^0 \gamma^\mu (D_\mu \Psi_m^0) - \overline{(D_\mu \Psi_m^0)} \gamma^\mu\, \Psi_m^0\Big)$, such that

$$\mathcal{L}_{class} = \overline{\Psi}_m^0 \big(\mathbb{I}\ (i\partial_\mu) + g\vec{T}.\vec{W}_\mu\big) \gamma^\mu \Psi_m^0 + \ldots \tag{33}$$

The T's are the (Cabibbo rotated) $SU(2)$ generators

$$T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & \mathcal{C}_0 \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline \mathcal{C}_0^\dagger & \end{array}\right), \tag{34}$$

[9] For example, in the $(\nu_\mu, \nu_\tau, \nu, \tau)$ sector, the condition writes $\left|\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{m_{\nu_\tau}^2 + m_{\nu_\mu}^2}\right| < \frac{g^2}{16\pi^2} \frac{m_\tau^2 - m_\mu^2}{m_W^2}$, the r.h.s. of which $\approx 1.9\,10^{-7}$, while the l.h.s. is experimentally known to be $\mathcal{O}(10^{-3})$ if one considers that the neutrino mass scale is $\mathcal{O}(eV)$. The mismatch is similar in the $(\nu_e, \nu_\tau, e, \tau)$ sector and worse in the (ν_e, ν_μ, e, μ) sector.

where $\mathcal{C}_0$ is the classical Cabibbo matrix

$$\mathcal{C}_0 = \mathcal{R}(\theta_c) = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} = \mathcal{C}_{u0}^\dagger\, \mathcal{C}_{d0} = \mathcal{R}(\theta_u)^\dagger\, \mathcal{R}(\theta_d). \tag{35}$$

Gauge currents and their $SU(2)_L$ algebra are thus directly related to kinetic terms by gauge invariance and the resulting Lagrangian is both gauge invariant and hermitian.

We shall use the same procedure to determine the Lagrangian after 1-loop transitions have been accounted for. Still in the bare mass basis Ψ_m^0, we have seen in subsection 2.2 that the kinetic terms, which are classically proportional, in momentum space, to $\mathbb{I}\,\not p$ get renormalized at 1-loop into $A(p^2, m_i, m_W)\,\not p$, with

$$A(p^2,\ldots) = \left(\begin{array}{c|c} K_u(p^2,\ldots) & \\ \hline & K_d(p^2,\ldots) \end{array}\right) = \mathbb{I} + \left(\begin{array}{c|c} H_u(p^2,\ldots) & \\ \hline & H_d(p^2,\ldots) \end{array}\right); \tag{36}$$

p_μ stands, there, for the common momentum of the ingoing and outgoing fermions, as depicted in Fig. 1. The 1-loop kinetic + gauge Lagrangian that we will hereafter consider is accordingly $i\,\overline{\Psi}_m^0\,\overleftrightarrow{AD_\mu}\,\gamma^\mu\Psi_m^0 \equiv \frac{i}{2}\Big(\overline{\Psi}_m^0\gamma^\mu(AD_\mu\,\Psi_m^0) - \overline{(AD_\mu\Psi_m^0)}\,\gamma^\mu\Psi_m^0\Big)$, which yields

$$\mathcal{L}_{1-loop} = \overline{\Psi}_m^0\left(A\,(i\partial_\mu) + \frac{g}{2}(A\vec{T} + \vec{T}A).\vec{W}_\mu\right)\gamma^\mu\Psi_m^0 + \ldots \tag{37}$$

It has the required properties of gauge invariance and, thanks to the presence of the symmetric expression $A\vec{T}+\vec{T}A$, of hermiticity (hermiticity is, instead, not achieved if one considers a kinetic Lagrangian of the form $i\,\overline{\Psi}_m^0\,\overrightarrow{AD_\mu}\,\gamma^\mu\Psi_m^0$ (with "$\rightarrow$" instead of "$\leftrightarrow$" on top of AD_μ)). Gauge invariance has in particular dictated the 1-loop expression of the gauge currents, from which we shall now deduce that of the 1-loop Cabibbo matrix.

4.2 The Cabibbo matrix $\mathfrak{C}(p^2,\ldots)$ stays unitary

The 1-loop Cabibbo matrix in the bare mass basis can be read directly from the expression $\frac{g}{2}\overline{\Psi}_m^0(A\vec{T} + \vec{T}A)\gamma^\mu\Psi_m^0$ of the gauge currents that results from (37). This yields

$$\mathcal{C}^{bm}(p^2,\ldots) = \frac{1}{2}\Big[\underbrace{(\mathbb{I}+H_u)}_{K_u(p^2,\ldots)}\,\mathcal{C}_0 + \mathcal{C}_0\,\underbrace{(\mathbb{I}+H_d)}_{K_d(p^2,\ldots)}\Big]. \tag{38}$$

A naive calculation could erroneously lead to the conclusion that $\mathcal{C}^{bm}$ is non-unitary. Indeed, using $\mathcal{C}_0 = \mathcal{R}(\theta_d - \theta_u)$ and the expressions (6) (7) for K_d and K_u, one finds $\mathcal{C}^{bm}(\mathcal{C}^{bm})^\dagger \neq \mathbb{I}$. However, these expressions are written in a basis which is non-orthogonal at 1-loop. Consider indeed, for example, the relation $\mathcal{C}_{11}^*\mathcal{C}_{12} + \mathcal{C}_{21}^*\mathcal{C}_{22} \neq 0$. It traduces the non-orthogonality of the two vectors $\mathcal{C}\begin{pmatrix}0\\1\end{pmatrix} \equiv \begin{pmatrix}\mathcal{C}_{12}\\ \mathcal{C}_{22}\end{pmatrix}$ and $\mathcal{C}\begin{pmatrix}1\\0\end{pmatrix} \equiv \begin{pmatrix}\mathcal{C}_{11}\\ \mathcal{C}_{21}\end{pmatrix}$ when their scalar product is evaluated with the metric $(1,1)$. However, this is the appropriate metric only at the classical level, where $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$, which represent fermions in bare mass space, are orthogonal since no transition occurs between the two of them; but it is no longer so at 1-loop (see Fig. 1) [10]. The pure kinetic terms in (37) are, in particular, not normalized to $\mathbb{I}$ but to the non-diagonal matrix A. It is thus necessary, before drawing any conclusion, to go to the orthogonal basis of 1-loop mass eigenstates by using the relation (23). Because of the unitarity of the $\mathcal{R}(\xi)$ rotations, one has $][\mathcal{V}_{u,d}\mathcal{R}(\xi_{u,d})]^\dagger K_{u,d}[\mathcal{V}_{u,d}\mathcal{R}(\xi_{u,d})] \equiv \mathcal{R}(\xi^\dagger_{u,d})[\mathcal{V}^\dagger_{u,d}K_{u,d}\mathcal{V}_{u,d}]\mathcal{R}(\xi_{u,d}) \stackrel{(1)}{=} \mathcal{R}(\xi_{u,d})^\dagger\mathcal{R}(\xi_{u,d}) = \mathbb{I}$, such that the pure kinetic terms get now normalized to $\mathbb{I}$. And, as we show next, the 1-loop Cabibbo matrix $\mathfrak{C}(p^2,\ldots)$ rewrites, then, as a rotation. It becomes indeed in this basis

$$\mathfrak{C}(p^2,\ldots) = [\mathcal{V}_u\,\mathcal{R}(\xi_u)]^\dagger\,\mathcal{C}^{bm}(p^2,\ldots)\,[\mathcal{V}_d\,\mathcal{R}(\xi_d)]. \tag{39}$$

Transforming the general expressions (39) and (38) with the help of (1) which entails $K_d = (\mathcal{V}_d^{-1})^\dagger\mathcal{V}_d^{-1}$ ($K_u = (\mathcal{V}_u^{-1})^\dagger\mathcal{V}_u^{-1}$), yields

$$\mathfrak{C} = \frac{1}{2}\mathcal{R}(\xi_u)^\dagger\left[\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d + \mathcal{V}_u^\dagger\mathcal{C}_0(\mathcal{V}_d^{-1})^\dagger\right]\mathcal{R}(\xi_d) = \frac{1}{2}\mathcal{R}(\xi_u)^\dagger\left[\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d + \left((\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d)^{-1}\right)^\dagger\right]\mathcal{R}(\xi_d). \tag{40}$$

Using the expression (18) for the $\mathcal{V}$'s, one gets

$\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d = \begin{pmatrix} \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_+^u}{1+t_+^d}} & \sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_+^u}{1+t_-^d}} \\ -\sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_-^u}{1+t_+^d}} & \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_-^u}{1+t_-^d}} \end{pmatrix}$ and $\left[\left(\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d\right)^{-1}\right]^\dagger =$ $\begin{pmatrix} \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_+^d}{1+t_+^u}} & \sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_+^d}{1+t_-^u}} \\ -\sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_-^d}{1+t_+^u}} & \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t_-^d}{1+t_-^u}} \end{pmatrix}$ which leads finally to

$$\mathfrak{C}(p^2,\ldots) = \mathcal{R}\Big(\big(\theta_d+\omega_d+\xi_d\big) - \big(\theta_u+\omega_u+\xi_u\big)\Big) + \mathcal{O}(g^{(>2)}).\ q.e.d. \tag{41}$$

$\mathfrak{C}(p^2)$ stays thus unitary for any common value of p^2 at which its entries are evaluated [11]. (41) shows that the Cabibbo angle $\theta_c = \theta_d - \theta_u$ gets renormalized by $(\omega_d+\xi_d)-(\omega_u+\xi_u)$.

In the basis of 1-loop mass eigenstates, the Lagrangian $\mathcal{L}$ rewrites

$$\mathcal{L} = \begin{pmatrix} \overline{u_{mL}} & \overline{c_{mL}} & \overline{d_{mL}} & \overline{s_{mL}} \end{pmatrix}(p^2,\ldots)\left(\not{p} + g\,\vec{\mathfrak{T}}(p^2,\ldots).\vec{W}_\mu\,\gamma^\mu + \ldots\right)\begin{pmatrix} u_{mL} \\ c_{mL} \\ d_{mL} \\ s_{mL}\end{pmatrix}(p^2,\ldots) + \ldots, \tag{42}$$

with "1-loop" $SU(2)_L$ generators $\vec{\mathfrak{T}}(p^2,\ldots)$ depending now on p^2 and on the masses

$$\mathfrak{T}^3(p^2,\ldots) = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), \mathfrak{T}^+(p^2,\ldots) = \left(\begin{array}{c|c} & \mathfrak{C}(p^2,\ldots) \\ \hline & \end{array}\right), \mathfrak{T}^-(p^2,\ldots) = \left(\begin{array}{c|c} & \\ \hline \mathfrak{C}^\dagger(p^2,\ldots) & \end{array}\right). \tag{43}$$

Our procedure has accordingly preserved the $SU(2)_L$ structure of gauge currents at 1-loop, which guarantees in particular that the corresponding Ward identities are satisfied.

We keep mentioning the dependence on p^2, reminding that it only goes away (becoming sub-leading in powers of $\frac{p^2}{m_W^2}$ when $p^2 \ll m_W^2$. Since we are not able to get the exact dependence on this variable, we

[10] Likewise, for any matrix U, the relation $UU^\dagger = 1$ traduces unitarity only if U is expressed in an orthogonal basis of states (*i.e.* no transition exists between them at the order that is considered).

[11] This may not be in contradiction with the non-unitarity claimed in [9] and [11] when the two external fermions legs are on different mass-shell, since, then, two different p^2 are involved. See also appendix A.1

shall keep on working in this approximation, which is only justified at energies well below the electroweak scale. Some remarks concerning the p^2 dependence are given in appendix A.

Note: One can easily demonstrate that $\mathfrak{C}(p^2,\ldots) = \mathcal{C}_u^\dagger\,\mathcal{C}_d + \mathcal{O}(g^2)$, reminiscent of the classical relation $\mathcal{C}_0 = \mathcal{C}_{u0}^\dagger\,\mathcal{C}_{d0}$, as follows. Since H_u and H_d in (38) are $\mathcal{O}(g^2)$, the terms proportional to them in (39) can be calculated with the expressions of $\mathcal{R}(\xi_d)$ and $\mathcal{V}_d$ at $\mathcal{O}(g^0)$, that is, for $t_+ = 0 = t_-$; one can accordingly take in there $\mathcal{R}(\xi_d) \stackrel{(22)}{\rightarrow} \mathcal{R}(-\omega_d)$ and $\mathcal{V}_d \stackrel{(18)}{\rightarrow} \mathcal{R}(\omega_d)$, such that $\mathcal{V}_d\mathcal{R}(\xi_d) \rightarrow \mathbb{I}$. The same approximation can be done in the (u,c) sector. The resulting expression for $\mathfrak{C}$ is

$$\mathfrak{C}(p^2,\ldots) \stackrel{\mathcal{O}(g^2)}{\approx} \mathcal{R}(\xi_u)^\dagger\,\mathcal{V}_u^\dagger\;\mathcal{C}_0\;\mathcal{V}_d\,\mathcal{R}(\xi_d) + \frac{1}{2}\big(\underbrace{H_u\,\mathcal{C}_0 + \mathcal{C}_0\,H_d}_{\mathcal{O}(g^2)}\big), \tag{44}$$

which leads to the announced formula after using (35), and (28) and its equivalent for $\mathcal{C}_u$. Since $\mathfrak{C}(p^2)$ is unitary, the non-unitarity of $\mathcal{C}_u^\dagger\mathcal{C}_d$ gets compensated by that of $\frac{1}{2}(H_u\mathcal{C}_0 + \mathcal{C}_0 H_d)$.

5 Restoring "perturbative stability": canceling non-diagonal transitions at 1-loop with counterterms

5.1 Instability close to degeneracy

Quasi-degenerate systems are known to be unstable with respect to small perturbations. This property is easily verified here, through the amount by which classical mixing angles are renormalized when 1-loop transitions are accounted for. It undergoes indeed large variations when the classical masses span a very small interval in the neighborhood of degeneracy: we first consider the case of exact classical degeneracy ($m_d = m_s$), secondly the pole of (30), which corresponds to a situation where d and s are extremely close to degeneracy (see subsection 3.2), and, last, the pole of $\tan 2\xi$, which also corresponds to quasi-degenerate fermions, but not as close as previously.

- For exact classical degeneracy $h_d = h_s$ such that, by the expression of $\sin 2\omega_d$ in (17), $\omega_d = -\theta_c$. (20) shows then that $\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d$ stays diagonal, and, so, $\xi_d = 0$ [12] . The classical (d,s) mixing angle θ_d is renormalized (see (29)) by $(\omega_d + \xi_d) = -\theta_c$ and becomes $\theta_d - \theta_c = \theta_u$, the classical mixing angle of the (u,c) pair.

According to (41), the Cabibbo mixing angle gets renormalized from its classical value θ_c to $\theta_c + (\omega_d + \xi_d) - (\omega_u + \xi_u) = -(\omega_u + \xi_u)$. This is vanishing by the equivalent of (16) which yields $\omega_u = 0$ for $h_d = h_s$, and then by that of (22) which entails $\xi_u = 0$ for $\omega_u = 0$. To such a system is accordingly associated a vanishing 1-loop Cabibbo angle. Renormalization effects can thus be large.

- At the pole of (30), by definition, the renormalization of θ_d becomes maximal ($\pm\frac{\pi}{4}$).
- At the pole of $\tan 2\xi_d$, it becomes instead minimally small (see subsection 3.2.1).

So, in a close neighborhood of degeneracy, the renormalization $(\omega_d+\xi_d)$ of θ_d undergoes large variations. So does the one of the Cabibbo angle.

5.2 The counterterms of Shabalin

Let us now add to the classical Lagrangian in bare mass space the counterterms which were first proposed by Shabalin in his study [6] of the electric dipole moment of quarks. They are devised to cancel the (p^2-dependent) $s_m^0 \leftrightarrow d_m^0$ transitions when either $p^2 = m_d^2$ or $p^2 = m_s^2$ (d or s on mass-shell). So, an on mass-shell s_m^0 cannot anymore transmute into a d_m^0 with the same virtuality, and *vice versa*. They were also introduced in [10] and [12]. In the short letter [12], the inclusion of these counterterms was proposed

[12]This is in agreement with (22) which shows that $\tan 2\xi_d$ has no pole when $m_d = m_s$.

as a solution to rescue the standard CKM phenomenology. In [10], only the classical Lagrangian + the counterterms were re-diagonalized, but the effective 1-loop transitions were not included. This completion is the goal of the lines below. We shall go through the same steps as previously, re-diagonalizing simultaneously the effective kinetic and mass terms up to $\mathcal{O}(g^2)$, including Shabalin's counterterms.

Following [10], let us accordingly add to the bare Lagrangian the kinetic and mass-like counterterms which concern both chiralities of fermions

$$-A_d\,\overline{d^0_m}\,\not p\,(1-\gamma^5)\,s^0_m - B_d\,\overline{d^0_m}\,(1-\gamma^5)\,s^0_m - E_d\,\overline{d^0_m}\,\not p\,(1+\gamma^5)\,s^0_m - D_d\,\overline{d^0_m}\,(1+\gamma^5)\,s^0_m. \tag{45}$$

Requesting that $s^0_m \to d^0_m$ transitions vanish when either s^0_m or d^0_m is on mass-shell yields (see Appendix A of [10])

$$\begin{aligned}
A_d &= s_c c_c \frac{m_d^2\,(h_u-h_c)_{p^2=m_d^2} - m_s^2\,(h_u-h_c)_{p^2=m_s^2}}{m_d^2-m_s^2} \approx s_c c_c\left((h_u-h_c)_{p^2=m_d^2} + m_s^2 \frac{\partial(h_u-h_c)}{\partial p^2}\Big|_{p^2=m_d^2}\right),\\
E_d &= s_c c_c \frac{m_s m_d\left((h_u-h_c)_{p^2=m_d^2} - (h_u-h_c)_{p^2=m_s^2}\right)}{m_d^2-m_s^2} \approx s_c c_c m_s m_d \frac{\partial(h_u-h_c)}{\partial p^2}\Big|_{p^2=m_d^2},\\
&\quad B_d = -m_s\,E_d, \quad D_d = -m_d\,E_d,
\end{aligned} \tag{46}$$

The re-diagonalization of the left-handed kinetic terms at 1-loop is operated via a non-unitary transformation $\mathcal{V}_d$ of the same form as (18). Counterterms only induce the replacement of $s_c c_c\,(h_u-h_c)(p^2,\ldots)$ with $s_c c_c\,(h_u-h_c)(p^2,\ldots) - A_d$, such that the angle ω_d changes from (16) to

$$\tan 2\omega_{dL}(p^2,\ldots) = \frac{-2\big(s_c c_c\,(h_u-h_c)(p^2,\ldots) - A_d\big)}{(h_u-h_c)(p^2,\ldots)\cos 2\theta_c + \left[\frac{(h_d-h_s)(p^2,\ldots)}{2}\right]}, \tag{47}$$

in which we have added a subscript "$_L$" to ω_d to distinguish it from its counterpart ω_{dR} associated with right-handed fermions.

The quantity $\big(s_c c_c\,(h_u-h_c)(p^2,\ldots) - A_d\big)$, which will be often encountered, writes

$$\begin{aligned}
s_c c_c\,(h_u-h_c)(p^2,\ldots) - A_d &\approx s_c c_c\left((h_u-h_c)(p^2,\ldots) - (h_u-h_c)_{p^2=m_d^2} - m_s^2\frac{\partial(h_u-h_c)}{\partial p^2}\Big|_{p^2=m_d^2}\right)\\
&\approx s_c c_c\left(p^2-(m_d^2+m_s^2)\right)\frac{\partial(h_u-h_c)}{\partial p^2}\Big|_{p^2=m_d^2},
\end{aligned} \tag{48}$$

in which we have taken $p^2 \sim m_d^2 \sim m_s^2$.

By differentiating (11) with respect to p^2, one gets, still in the limit $p^2, m_i^2, m_j^2 \ll m_W^2$ and in the $\overline{MS}$ scheme

$$\frac{\partial(h_i-h_j)}{\partial p^2} \underset{\overline{MS}}{\overset{p^2,m_i^2,m_j^2\ll m_W^2}{\approx}} 3\,\frac{g^2}{4}\frac{i}{16\pi^2}\frac{m_i^2-m_j^2}{m_W^4}. \tag{49}$$

One has now (we added a superscript "d" to t_+ and t_- because $A_d \neq A_u$, such that $t^d_+ \neq t^u_+, t^u_- \neq t^d_-$, and also a subscript "$_L$" to recall that they concern left-handed fields)

$$\begin{aligned}
t^d_{\pm L}(p^2,\ldots) &= \frac{h_u+h_c+\left[\frac{h_d+h_s}{2}\right]}{2}(p^2,\ldots)\\
&\pm\frac{1}{2}\sqrt{\left((h_u-h_c)(p^2,\ldots)\cos 2\theta_c + \left[\frac{(h_d-h_s)(p^2,\ldots)}{2}\right]\right)^2 + 4\big(s_c c_c\,(h_u-h_c)(p^2,\ldots) - A_d\big)^2},
\end{aligned} \tag{50}$$

which gives back (14) when A_d is set to zero.

As far as the right-handed kinetic terms are concerned, they are controlled by the matrix $\begin{pmatrix} 1 & -E_d \\ -E_d & 1 \end{pmatrix}$ and are accordingly re-diagonalized into the unit matrix by a non-unitary transformation $\mathcal{U}_d$

$$\mathcal{U}_d^\dagger \begin{pmatrix} 1 & -E_d \\ -E_d & 1 \end{pmatrix} \mathcal{U}_d = \mathbb{I}, \quad \mathcal{U}_d = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{1+E_d}} & \frac{1}{\sqrt{1-E_d}} \\ \frac{-1}{\sqrt{1+E_d}} & \frac{1}{\sqrt{1-E_d}} \end{pmatrix} \Rightarrow \mathcal{U}_d\mathcal{U}_d^\dagger = \frac{1}{1-E_d^2} \begin{pmatrix} 1 & E_d \\ E_d & 1 \end{pmatrix}. \tag{51}$$

It corresponds to $\omega_{dR} = \frac{\pi}{4}, t^d_{+R} = E_d, t^d_{-R} = -E_d$.

The mass matrix to diagonalize is now $\mathcal{V}_d^\dagger M_d \mathcal{U}_d$, where, including the counterterms, M_d is now given by

$$M_d = \begin{pmatrix} m_d & D_d \equiv -m_d E_d \\ B_d \equiv -m_s E_d & m_s \end{pmatrix}. \tag{52}$$

The rotation $\mathcal{R}(\xi_{dL})$ will accordingly diagonalize the matrix $(\mathcal{V}_d^\dagger M_d \mathcal{U}_d)(\mathcal{U}_d^\dagger M_d^\dagger \mathcal{V}_d)$.

Neglecting irrelevant terms proportional to $E^{\geq 2}$ and to $g^{>2}$, one gets

$$\begin{aligned}
\mathcal{V}_d^\dagger M_d \mathcal{U}_d \mathcal{U}_d^\dagger M_d^\dagger \mathcal{V}_d \;&=\; \begin{pmatrix} \frac{m_d^2 c^2_{\omega_{dL}} + m_s^2 s^2_{\omega_{dL}} + 4 m_d m_s E_d s_{\omega_{dL}} c_{\omega_{dL}}}{1+t^d_{+L}} & \frac{(m_d^2 - m_s^2) s_{\omega_{dL}} c_{\omega_{dL}} - 2 m_d m_s E_d (c^2_{\omega_{dL}} - s^2_{\omega_{dL}})}{\sqrt{(1+t^d_{+L})(1-t^d_{-L})}} \\ \frac{(m_d^2 - m_s^2) s_{\omega_{dL}} c_{\omega_{dL}} - 2 m_d m_s E_d (c^2_{\omega_{dL}} - s^2_{\omega_{dL}})}{\sqrt{(1+t^d_{+L})(1-t^d_{-L})}} & \frac{m_d^2 s^2_{\omega_{dL}} + m_s^2 c^2_{\omega_{dL}} - 4 m_d m_s E_d s_{\omega_{dL}} c_{\omega_{dL}}}{1+t^d_{-L}} \end{pmatrix} \\
&+\; m_d m_s E_d \begin{pmatrix} -\sin 2\omega_{dL} & \cos 2\omega_{dL} \\ \cos 2\omega_{dL} & \sin 2\omega_{dL} \end{pmatrix}.
\end{aligned} \tag{53}$$

The expression (22) for $\tan 2\xi_d$ gets replaced by

$$\tan 2\xi_{dL}(p^2, \ldots) = \frac{-(m_d^2 - m_s^2)\sin 2\omega_{dL} + 2 m_d m_s E_d \cos 2\omega_{dL}}{(m_d^2 - m_s^2)\cos 2\omega_{dL} + 2 m_d m_s E_d \sin 2\omega_{dL} - (m_d^2 + m_s^2)\frac{t^d_{+L} - t^d_{-L}}{2}}, \tag{54}$$

in which we have neglected factors $(1 + \alpha t^d_{+L} + \beta t^d_{-L}), \alpha, \beta = \mathcal{O}(1)$, which yield contributions of unwanted higher order in g.

Unless $\cos 2\theta_c \approx -\frac{1}{2}\frac{h_d - h_s}{h_u - h_c} \stackrel{(13)}{\approx} -\frac{m_d^2 - m_s^2}{m_u^2 - m_c^2}$, (47), (48) and (49), show that, when $p^2 \ll m_W^2$ and since $m_u^2, m_c^2 \ll m_W^2$, $\omega_{dL} \sim m_s^2/m_W^2$ is very small. Then, using $\sin 2\omega_{dL} \approx \tan 2\omega_{dL}$, the expression for E_d in (46) and the one for $t^d_{+L} - t^d_{-L}$ coming from (50) (in which we neglect the term $4(s_c c_c (h_u - h_d) - A_d)$), (54) rewrites (the term $2 m_d m_s E_d \sin 2\omega_{dL}$ in its denominator can always be neglected)

$$\tan 2\xi_{dL} \approx 2 s_c c_c \frac{\partial (h_u - h_c)}{\partial p^2} \left(\frac{(m_d^2 - m_s^2)\left(p^2 - (m_d^2 + m_s^2)\right) + m_d^2 m_s^2}{(m_d^2 - m_s^2) - \frac{m_d^2 + m_s^2}{2}\left((h_u - h_c)\cos 2\theta_c + \left[\frac{h_d - h_s}{2}\right]\right)} \right), \tag{55}$$

showing, with (49), that $\xi_{dL} \sim (p^2, m^2)/m_W^2$ is also very small.

When $\cos 2\theta_c \approx -\frac{1}{2}\frac{h_d - h_s}{h_u - h_c} \stackrel{(13)}{\approx} -\frac{m_d^2 - m_s^2}{m_u^2 - m_c^2}$, $\tan 2\omega_{dL} \to \infty$, which corresponds to ω_{dL} maximal. Then, (54) and (50) yield $\tan 2\xi_{dL} \to -\frac{m_d^2 - m_s^2}{2 m_d m_s E_d - (m_d^2 + m_s^2)\left(s_c c_c (h_u - h_d) - A_d\right)}$, which, using (46) and (48), is finally equivalent to $\tan 2\xi_{dL} = -\frac{m_d^2 - m_s^2}{s_c c_c \frac{\partial (h_u - h_c)}{\partial p^2}} \frac{1}{2 m_d^2 m_s^2 - (m_d^2 + m_s^2)\left(p^2 - (m_d^2 + m_s^2)\right)}$. Unless d and s are exactly degenerate (in which case ξ_{dL} shrinks to 0), this yields a quasi-maximal ξ_{dL}, because of the very small value of $\frac{\partial (h_u - h_c)}{\partial p^2}$, given in (49).

This is however not true when the numerator of (47) vanishes, which occurs for $s_c c_c(h_u - h_c) - A_d = 0$, or, likewise, by (50), for $t^+_{dL} = t^-_{dL}$. In this case, ω_{dL} is undetermined and can be taken to vanish, since the matrix of kinetic terms is proportional to the unit matrix. One then finds a very small $\tan 2\xi_{dL} = \frac{2m_d m_s E_d}{m_d^2 - m_s^2}$ (see (46) and (49)).

The expressions obtained in the (u,c) channel are very similar. One gets:

$$
\begin{aligned}
A_u &= -s_c c_c \frac{m_u^2\,(h_d - h_s)_{p^2=m_u^2} - m_c^2\,(h_d - h_s)_{p^2=m_c^2}}{m_u^2 - m_c^2} \approx -s_c c_c \left((h_d - h_s)_{p^2=m_u^2} + m_c^2 \frac{\partial (h_d - h_s)}{\partial p^2}\Big|_{p^2=m_u^2} \right);\\
E_u &= -s_c c_c \frac{m_u m_c \left((h_d - h_s)_{p^2=m_u^2} - (h_d - h_s)_{p^2=m_c^2} \right)}{m_u^2 - m_c^2} \approx s_c c_c m_s m_d \frac{\partial (h_d - h_s)}{\partial p^2}\Big|_{p^2=m_u^2};\\
&\quad B_u = -m_c\, E_u, \quad D_u = -m_u\, E_u;
\end{aligned}
\tag{56}
$$

$$
\tan 2\omega_{uL}(p^2,\ldots) = \frac{-2\big(-s_c c_c\,(h_d - h_s)(p^2,\ldots) - A_u\big)}{(h_d - h_s)(p^2,\ldots)\cos 2\theta_c + \left[\frac{(h_u - h_c)(p^2,\ldots)}{2}\right]}; \tag{57}
$$

$$
\begin{aligned}
-s_c c_c\,(h_d - h_s)(p^2,\ldots) - A_u &\approx -s_c c_c \left((h_d - h_s)(p^2,\ldots) - (h_d - h_s)_{p^2=m_u^2} - m_c^2 \frac{\partial (h_d - h_s)}{\partial p^2}\Big|_{p^2=m_u^2} \right)\\
&\approx -s_c c_c \left(p^2 - (m_u^2 + m_c^2)\right) \frac{\partial (h_d - h_s)}{\partial p^2}\Big|_{p^2=m_u^2};
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
t^u_{\pm L}(p^2,\ldots) &= \frac{\left[\frac{h_u + h_c}{2}\right] + h_d + h_s}{2}(p^2,\ldots)\\
&\pm \frac{1}{2}\sqrt{\left((h_d - h_s)(p^2,\ldots)\cos 2\theta_c + \left[\frac{(h_u - h_c)(p^2,\ldots)}{2}\right]\right)^2 + 4\big(-s_c c_c\,(h_d - h_s)(p^2,\ldots) - A_u\big)^2};
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
\tan 2\xi_{uL}(p^2,\ldots) &= \frac{-(m_u^2 - m_c^2)\sin 2\omega_{uL} + 2 m_u m_c E_u \cos 2\omega_{uL}}{(m_u^2 - m_c^2)\cos 2\omega_{uL} + 2 m_u m_c E_u \sin 2\omega_{uL} - (m_u^2 + m_c^2)\frac{t^u_{+L} - t^u_{-L}}{2}}\\
&\approx -2 s_c c_c \frac{\partial (h_d - h_s)}{\partial p^2} \left(\frac{(m_u^2 - m_c^2)\left(p^2 - (m_u^2 + m_c^2)\right) + m_u^2 m_c^2}{(m_u^2 - m_c^2) - \frac{m_u^2 + m_c^2}{2}\left((h_d - h_s)\cos 2\theta_c + \left[\frac{h_u - h_c}{2}\right]\right)} \right).
\end{aligned}
\tag{60}
$$

Unlike in the (d,s) sector, because $|m_d - m_s| < |m_u - m_c|$, $\tan 2\omega_{ul}$ given by (57) cannot have any pole. This makes ω_{uL} always very small and, likewise, ξ_{uL}. Furthermore, the equality $t^+_{uL} = t^-_{uL}$ can never be achieved (see also section 6). These results stay true when $m_d = m_s$, in which case $h_d = h_s$, which entails that A_u, E_u, B_u, D_u, ω_{uL} and ξ_{uL} vanish.

5.3 Stability is restored

We now check that Shabalin's counterterms stabilize 1-loop mixing angles in the vicinity of $d - s$ degeneracy.

Still except when $\cos 2\theta_c = -\frac{1}{2}\frac{h_d - h_s}{h_u - h_c}$, which corresponds, when $m_d = m_s$, to θ_c maximal (see also subsection 6.2), ω_{dL} stays small when $m_d \approx m_s$. From (47), (48), (49), one gets

$$
\tan 2\omega_{dL} \underset{\overline{MS}}{\overset{p^2, m_d^2 \sim m_s^2, m_u^2, m_c^2 \ll m_W^2}{\approx}} -3\,\frac{p^2 - 2m_d^2}{m_W^2} \tan 2\theta_c, \tag{61}
$$

and so does ξ_{dL}, which, from (55), becomes

$$\tan 2\xi_{dL} \overset{p^2, m_d^2=m_s^2, m_u^2, m_c^2 \ll m_W^2}{\underset{\overline{MS}}{\approx}} -3\,\frac{m_d^2}{m_W^2}\frac{1}{\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)}\tan 2\theta_c, \tag{62}$$

since, for $\mu^2 \in [m_K^2, m_D^2]$, $\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) \in [7, 12]$.
So, when $m_d \simeq m_s$, the mixing angle θ_{dL} is accordingly renormalized at 1-loop by the small quantity $\omega_{dL}+\xi_{dL} \approx \frac{1}{2}(\tan 2\omega_{dL} + \tan 2\xi_{dL}) \sim \frac{m_d^2}{m_W^2}\tan 2\theta_c$.
In the (u,c) sector, $E_u = 0 = A+u$ when $m_d = m_s$ and one gets

$$\tan 2\xi_u \approx -\tan 2\omega_{uL} = -\frac{4A_u}{h_u - h_c} = 0, \tag{63}$$

such that θ_{uL} is not renormalized at all.

5.4 The Cabibbo matrix $\mathfrak{C}(p^2,\ldots)$ still stays unitary

The expression for $\mathfrak{C}(p^2,\ldots)$ is still given by (40), but one must now accounts for $t^u_{\pm L} \neq t^d_{\pm L}$ since $A_u \neq A_d$. One gets now $\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d = \begin{pmatrix} \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^u_{+L}}{1+t^d_{+L}}} & \sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^u_{+L}}{1+t^d_{-L}}} \\ -\sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^u_{-L}}{1+t^d_{+L}}} & \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^u_{-L}}{1+t^d_{-L}}} \end{pmatrix}$ and $\left[\left(\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d\right)^{-1}\right]^\dagger = \begin{pmatrix} \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^d_{+L}}{1+t^u_{+L}}} & \sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^d_{+L}}{1+t^u_{-L}}} \\ -\sin(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^d_{-L}}{1+t^u_{+L}}} & \cos(\theta_c-\omega_u+\omega_d)\sqrt{\frac{1+t^d_{-L}}{1+t^u_{-L}}} \end{pmatrix}$, which leads to the same formula (41) as before for $\mathfrak{C}(p^2,\ldots)$, which is unitary. Accordingly, like in the absence of Shabalin's counterterms, the classical Cabibbo angle θ_c gets renormalized at 1-loop by
$(\omega_{dL}+\xi_{dL})(p^2, m_d^2, m_s^2, m_u^2, m_c^2, m_W^2) - (\omega_{uL}+\xi_{uL})(p^2, m_d^2, m_s^2, m_u^2, m_c^2, m_W^2)$.
For more remarks concerning the p^2 dependence, see appendix A.

6 Suppressing extra flavor changing neutral currents

The absence of flavor changing neutral currents is classically implemented *ab initio* in bare flavor space by the canonical choice of the kinetic terms, proportional to the unit matrix, and by that of the $SU(2)_L$ generators which, in the (u,c,d,s) basis, write $T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & 1 \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline 1 & \end{array}\right)$. The diagonality of the T^3 generator ensures that the W^3 gauge boson only couples, in both (u,c) and (d,s) sectors, to diagonal fermionic currents: no FCNC occurs classically. That this property is preserved in bare mass space is the essence of the GIM mechanism: the closure of the $SU(2)_L$ algebra (34) on the same T^3 as above is ensured by the unitarity of the classical Cabibbo matrix $\mathcal{C}_0$. The situation is different at 1-loop since vertex corrections with an internal charged gauge boson induce non-diagonal couplings of the W^3 gauge field (see Fig. 1 left) and also, for example, the non-diagonal $s \to d$ transition of Fig. 2 inserted on one of the two external fermion legs of a $W^3 s\bar{s}$ vertex triggers: – 1-loop FCNC's if one considers $s^0_f \to d^0_f$ transitions, – their equivalent for mass states if one considers, like we did, $s^0_m \to d^0_m$ transitions (see Fig. 1 right).

We have seen with (43), and this stays valid in the presence of Shabalin's counterterms, that, in the 1-loop mass basis, the $SU(2)_L$ algebra closes on the "canonical" $\mathfrak{T}^3 \equiv T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right)$. So, after 1-loop transitions of the type of Fig. 2, have been accounted for, one is back to a situation similar to the classical one. 1-loop non-diagonal neutral gauge currents are triggered by vertex corrections. As for the second origin of FCNC, insertion of Fig. 2 on one of the external leg of a $W^2 f\bar f$ vertex (Fig. 1 right), it is important to recall, as was demonstrated in [10] (Appendix B), that the introduction of Shabalin's counterterms do not modify transitions of the type $s \to dW^3$: the counterterms do cancel the non-diagonal transitions on external legs, but $s \to dW^3$ transitions are re-created with the same amplitude through the covariant derivative that has to be used inside them.

Is the situation strictly identical to the standard one? The answer is "not exactly", and this is what we investigate now. The issue is that of the existence of mass splittings, which are responsible for two facts:
* the slight non-unitarity of the connection between the orthogonal set of 1-loop mass eigenstates and bare mass (or flavor) states;
* that the two fermions concerned by 1-loop non-diagonal transitions (Fig. 2) cannot be both on mass-shell, such that Shabalin's counterterms can only restore 1-loop orthogonality between one on mass-shell fermion and a second one which is off mass-shell.

Since, by construction, 1-loop mass eigenstates as we defined them, by the diagonalization of the 1-loop quadratic effective Lagrangian (kinetic + mass terms), are orthogonal, the non-unitarity of their connection to bare mass states (and, thus, to bare flavor states, since the last two are unitarily connected) makes FCNC still occur in bare flavor (or mass) space. This trivially appears by transforming back the $W^3 f\bar f$ coupling in the space of 1-loop mass states, that we emphasized to be "canonical" (proportional to T^3), to bare flavor space. So, we face a situation where, because of (unavoidable) mass splittings, the standard situation in bare flavor space is spoiled.

We adopt a conservative point of view, require that the phenomenology should not differ from the standard one, and therefore that these extra FCNC vanish or, at least, are strongly damped.

6.1 When no counterterm is added

As soon as 1-loop transitions Fig. 2 are accounted for, the bare flavor (or mass) states do not form anymore an orthogonal set, such that requesting the absence of FCNC in this basis appears somewhat academic. In spite of this, and since the principle of the method and formulae will keep valid when counterterms are introduced, we proceed with this first case.

To that purpose, it is enough to use the relation (27) between 1-loop mass eigenstates and bare flavor states (and its equivalent in the (u,c) sector), which leads to the expression (28) for the 1-loop mixing matrix $\mathcal{C}_d$. Neutral gauge currents in the space of 1-loop mass eigenstates being proportional to T^3, their expression in bare flavor space gets simply proportional to $(\mathcal{C}_d^{-1})^\dagger \mathcal{C}_d^{-1} = (\mathcal{C}_d \mathcal{C}_d^\dagger)^{-1} \stackrel{(28)}{=} (\mathcal{C}_{d0}\mathcal{V}_d\mathcal{V}_d^\dagger\mathcal{C}_{d0}^\dagger)^{-1}$, and a similar expression in the (u,c) sector. From the expression (18) of $\mathcal{V}_d$, it is easy matter to get (T_x is defined in (5))

$$\begin{aligned} \mathcal{C}_{d0}\mathcal{V}_d\mathcal{V}_d^\dagger\mathcal{C}_{d0}^\dagger &= \frac{1}{(1+t^d_{+L})(1+t^d_{-L})}\Big(1+\frac{t^d_{+L}+t^d_{-L}}{2} - (t^d_{+L}-t^d_{-L})\mathcal{T}_x\big(-2(\theta_{dL}+\omega_{dL})\big)\Big) \\ \Rightarrow (\mathcal{C}_{d0}\mathcal{V}_d\mathcal{V}_d^\dagger\mathcal{C}_{d0}^\dagger)^{-1} &\approx (1+t^d_{+L})(1+t^d_{-L})\Big(1-\frac{t^d_{+L}+t^d_{-L}}{2} + (t^d_{+L}-t^d_{-L})\mathcal{T}_x\big(-2(\theta_{dL}+\omega_{dL})\big)\Big), \end{aligned} \tag{64}$$

which makes FCNC's proportional to $-(t^d_{+L}-t^d_{-L})\sin 2(\theta_{dL}+\omega_{dL})$ (the sine function corresponds to the non-diagonal terms of $\mathcal{T}_x$, as it appears in (5)), and an equivalent expression in the (u,c) sector. According to (64), in both the (d,s) and (u,c) sectors, their suppression requires that $(t^{u,d}_{+L}-t^{u,d}_{-L})\sin 2(\theta_{uL,dL}+\omega_{uL,dL})$ vanishes or, at least, that it be as small as possible.

• According to (14), the equality of t^d_{+L} and t^d_{-L} requires
$\cos 2\theta_c = -\frac{1}{2}\left(\frac{h_u-h_c}{h_d-h_s}+\frac{h_d-h_s}{h_u-h_c}\right) \approx -\frac{1}{2}\left(\frac{m_c^2-m_u^2}{m_s^2-m_d^2}+\frac{m_s^2-m_d^2}{m_c^2-m_u^2}\right)$. This corresponds to $|\cos 2\theta_c| > 1$, which can never be satisfied.

• FCNC's can accordingly only be suppressed if $(\omega_{dL}+\theta_{dL}) \approx 0$ and an equivalent condition in the (u,c) sector. As already mentioned in subsection 2.3, when (d,s) are much closer to degeneracy that (u,c), $\omega_{dL} \approx -\theta_c$ such that the condition for FCNC suppression rewrites $\theta_{dL} \approx \theta_c$. One also finds that $\theta_{uL} \approx -\omega_{uL}$ becomes small (see subsection 5.3). So, FCNC's get suppressed when bare flavor and mass states for the fermion pair which is the farthest from degeneracy get close to alignment. No condition on θ_c arises in this case.

6.2 In the presence of Shabalin's counterterms

If bare flavor states were a set of truly orthogonal states at 1-loop, they could only be unitarily connected with 1-loop mass eigenstates since the latter are constructed as being orthogonal. Then, the absence of FCNC would naturally translate from one basis to the other. That, instead, non-unitarity persists even in the presence of counterterms can be traced out in the expression (18) for $\mathcal{V}_d$, to the relations (19), and is due to $t^+_{dL} \neq t^-_{dL}$.
Relations (64) keep valid such that the discussion stays formally the same as in subsection 6.1). Results are different because the expression of ω_{dL} has changed into (47); so has the formula for $t_\pm$ which is now given by (50). Unlike previously, maximal mixing turns out to be one of the two types of solutions that arise.

• While, in the absence of counterterms, neither $t^+_{dL} = t^-_{dL}$, nor $t^+_{uL} = t^-_{uL}$ could be satisfied, in their presence the first relation now can be. According to (50), the equality of t^d_{+L} and t^d_{-L} requires both $\cos 2\theta_c = -\frac{1}{2}\frac{h_d-h_s}{h_u-h_c} \approx -\frac{1}{2}\frac{m_d^2-m_s^2}{m_u^2-m_c^2}$ and $(s_c c_c(h_u-h_c)-A_d)=0$. This corresponds to a Cabibbo angle close to maximal and, according to (48), to $p^2 = m_d^2+m_s^2$. At these values of θ_c and p^2, the 1-loop kinetic terms for the d-type fermions become proportional to $\left(1+\frac{h_u+h_c+\left[\frac{h_d+h_s}{2}\right]}{2}\right)\mathbb{I}$, making ω_{dL} undetermined. It can be in particular taken to vanish, such that, according to (54), ξ_{dL} is then very small.

In the (u,c) channel, since $(m_c-m_u) > (m_s-m_d)$, one can never have $t^u_{+L} = t^u_{-L}$ because this would correspond to $|\cos 2\theta_c| > 1$. So, FCNC's can only be suppressed, there, for $\theta_{uL} = -\omega_{uL}(p^2,\ldots)$. Strictly speaking, since θ_{uL} is a constant and ω_{uL} a function of p^2 and of the masses, the equality can only take place at one value of p^2. However, since all dependence's on p^2 are always very weak, $(\theta_{uL}+\omega_{uL})$ will only deviate very little from zero when p^2 varies. Since $(-s_c c_c(h_d-h_s)-A_u)$ is always very small, the equivalent of (47) entails that so is $\omega_{uL}(p^2,\ldots)$, and, by the equivalent of (54), so is $\xi_{uL}(p^2,\ldots)$.

The set $(t^d_{+L} = t^d_{-L}, \theta_{uL} = -\omega_{uL})$ constitutes the first possibility to suppress FCNC's at 1-loop. It corresponds to a quasi-maximal Cabibbo angle, to small θ_{uL}, small ω_{uL}, to $\omega_{dL}=0$ and to small ξ_{dL}. Accordingly, θ_{dL} is also quasi-maximal, and all angles get renormalized at 1-loop by small quantities, which makes this solution perturbatively safe. Note that, since θ_{uL} is small and stays so at 1-loop, this corresponds to a quasi-alignment of flavor and mass states in the channel with the largest mass splitting.

For the same θ_c (close to maximal) but when $p^2 \neq m_d^2+m_s^2$, $(s_c c_c(h_u-h_c)-A_d)$ stays very small (see (48), (49)). $\tan 2\omega_{dL}$ given by (47) becomes infinite, which corresponds to ω_{dL} maximal. The FCNC's can be taken to vanish (neglecting a very weak dependence on p^2) for $\theta_{dL} = -\omega_{dL}$, which is then maximal, too (like in the previous case). θ_{dL} gets renormalized at 1-loop into $\theta_{dL}+\omega_{dL}+\xi_{dL} = \xi_{dL}$ such that $\tan 2\xi_{dL} \stackrel{(54)}{\approx} -\frac{m_d^2-m_s^2}{2m_d m_s E_d-(m_d^2+m_s^2)\frac{f_d(p^2,\ldots)-A_d}{2}}$, which is very large. So, ξ_{dL} becomes close to maximal, too. This makes the classical and maximal θ_{dL} renormalized by a small amount, which however results from the cancellation between two large angles. In the (u,c) channel, things are like previously: small $\theta_{uL} = -\omega_{uL}$, and small ξ_{uL}.

This case is thus similar to the previous one in the sense that θ_c has the same large value, θ_{dL} too, that θ_{uL} is small, and that all of them are renormalized at 1-loop by small quantities. However, that the renormalization of θ_{dL} results from the cancellation between two large angles raises the question whether this situation is perturbatively safe. The answer is positive for two reasons:
* a small variation in p^2 away from $(m_d^2 + m_s^2)$, that is outside any of the two concerned mass-shells, is not expected to change the nature of the perturbative series;
* the 1-loop calculation that we performed in the bare mass basis can as well be done in the bare flavor basis; since the two are related by a unitary transformation $\mathcal{R}(\theta_{dL})$, such a transformation cannot change either the character of the perturbative series. Going through the same steps, one easily finds that ω_{dL} gets replaced by $(\omega_{dL} + \theta_{dL})$, which is now very small. In the bare flavor basis, one finds that the maximal θ_{dL} still gets, of course, renormalized by a small amount, but this now results from the sum of two small quantities, which is a perturbatively safe situation.

• Like in the absence of counterterms, from (64), FCNC's can also be canceled when the two conditions, respectively $\theta_{dL} = -\omega_{dL}(p^2, \ldots)$ in the (d, s) channel, and $\theta_{uL} = -\omega_{uL}(p^2, \ldots)$ in the (u, c) channel, are satisfied (or very close to this, because of the very weak dependence on p^2), without, now, any relation connecting $(t^d_{+L} - t^d_{-L})$ and θ_c. Then, since, for $p^2, m^2 < m_W^2$, $(s_c c_c(h_u - h_d) - A_d)$ and $(s_c c_c(h_d - h_s) - A_u)$ are small, so are $\omega_{uL,dL}(p^2, \ldots)$ and $\xi_{uL,dL}(p^2, \ldots)$. Accordingly, θ_{uL} and θ_{dL} are both small and renormalized at 1-loop by small quantities. This corresponds to a small θ_c, which is also renormalized by a small quantity. This configuration is perturbatively safe.

This discussion can be straightforwardly transposed to the leptonic case.

In addition to stabilizing the 1-loop renormalization of mixing angles in the vicinity of degeneracy, the introduction of Shabalin's counterterms has been seen to promote maximal mixing (in one channel, accompanied with quasi-alignment in the other channel) as one of the two natural solutions to the suppression of extra FCNC in the bare flavor basis. Maximal mixing cannot play this role in their absence.

A delicate issue is of course to discriminate between the two types of solutions, and to determine why one or the other should be preferred. Since $t_{\pm L}$ are the eigenvalues of the 1-loop kinetic terms, the equality $t_{+L} = t_{-L}$ corresponds to the case where, up to an overall renormalization $\frac{1}{\sqrt{1+t_\pm}}$, they can be re-diagonalized by a unitary $\mathcal{V}$ (see (18)); in the corresponding channel, which corresponds to the fermionic pair the closest to degeneracy, the individual mixing matrix $[\mathcal{C}_{0d}\mathcal{V}_d\mathcal{R}(\xi_d)](p^2, \ldots)$ becomes unitary, too (such that, in addition to the suppression of FCNC, neutral gauge currents also satisfy the property of universality). A quasi-maximal Cabibbo (or PMNS) angle corresponds to a minimization of FCNC's, to the smallest possible deviation from unitarity of the individual mixing matrix in the channel which is the closest to degeneracy, to a quasi-maximal individual mixing in this same channel, and to the quasi-alignment of flavor and mass eigenstates in the other channel. This situation corroborates a common argumentation that mass and flavor eigenstates of charged leptons coincide [13].

In the quark sector, reversely, the distinction between the two types of fermions, both charged, and which, furthermore, are not observed as particles, is less clear. The second solution to the suppression of FCNC's, in which both mixing angles are small, and which treats the two channels on an equal footing, looks then more adapted to the situation.

Note that the landscape that we obtain in this work is similar to the one present in [10]. Two types of solutions to the unitarization equation were uncovered there: the so-called "Cabibbo-like" solutions, in which no constraint occurred for the Cabibbo angle, and maximal mixing. The Cabibbo angle could then only be constrained by additional assumption; it turned out, there, that a suitable one was that universality and the absence of FCNC were violated with the same strength.

7 The case of 3 generations

Our goal is now to generalize the previous calculations to the case of 3 generations of fermions, asking in particular that no extra (with respect to the "standard" phenomenology) FCNC is present at 1-loop in the basis of bare flavor states in the presence of Shabalin's counterterms.

A major difference with the case of two generations is, in the quark sector, the presence of the heavy top quark $m_t \simeq 2m_W$. This makes in particular invalid the approximation $m \ll m_W$ for all fermion masses m, that we used in this case.

7.1 Conditions for suppressing extra FCNC (in the presence of counterterms)

Like in the case of two generations, extra FCNC will be absent in the (d,s,b) sector iff $\mathcal{C}_{d0}\mathcal{V}_d\mathcal{V}_d^\dagger\mathcal{C}_{d0}^\dagger = diag(\alpha^d,\beta^d,\gamma^d)$ diagonal. (not necessarily proportional to the unit matrix), where $\mathcal{C}_{d0}$ represents now the 3×3 classical mixing matrix for (d,s,b) quarks. Similar expressions occur in the (u,c,t) sector and for the two leptonic ones.

K_d being the kinetic terms of (d,s,b) at 1-loop (eventually including Shabalin's counterterms), (1) $\Rightarrow$ $\mathcal{V}_d\mathcal{V}_d^\dagger = K_d^{-1}$, such that $K_d^{-1} = \mathcal{C}_{d0}^\dagger\, diag(\alpha^d,\beta^d,\gamma^d)\,\mathcal{C}_{d0}$. Now, Shabalin's counterterms are precisely devised so as to (nearly, that is, up to a very weak dependence in p^2) cancel non-diagonal terms in K_d, which originate from 1-loop transitions of the type depicted in Fig. 2. Accordingly, in their presence, K_d, and thus K_d^{-1}, too, are practically diagonal. The condition for suppressing extra FCNC rewrites accordingly $\mathcal{C}_{d0}^\dagger\, diag(\alpha^d,\beta^d,\gamma^d)\,\mathcal{C}_{d0} = diagonal$. and we insist that it is only valid in the presence of counterterms.

Since $\mathcal{C}_{d0}$ is unitary, the condition rewrites: $\alpha^d\mathbb{I} + \mathcal{C}_{d0}^\dagger\, diag(0, u^d \equiv \beta^d-\alpha^d, v^d\equiv \gamma^d-\alpha^d)\,\mathcal{C}_{d0}$ diagonal. The first term, proportional to α^d, being already diagonal, the condition applies to the second contribution. Forgetting, as we always did, about CP violating phases, it is convenient to parametrize $\mathcal{C}_{d0} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, with $\mathcal{R}_{23} = \begin{pmatrix} 1 & 0 & 0\\ 0 & c_{23}^d & s_{23}^d \\ 0 & -s_{23}^d & c_{23}^d\end{pmatrix}$, $\mathcal{R}_{13} = \begin{pmatrix} c_{13}^d & 0 & s_{13}^d\\ 0 & 1 & 0 \\ -s_{13}^d & 0 & c_{13}^d\end{pmatrix}$, $\mathcal{R}_{12}$ $\begin{pmatrix} c_{12}^d & s_{12}^d & 0\\ -s_{12}^d & c_{12}^d & 0 \\ 0 & 0 & 1\end{pmatrix}$, to search for eventual solutions different from $\alpha^d=\beta^d=\gamma^d$ ($u^d = 0 = v^d$). Equating to zero the 3 non-diagonal entries of the symmetric matrix $\mathcal{C}_{d0}^\dagger\, diag(\alpha^d,\beta^d,\gamma^d)\,\mathcal{C}_{d0}$ yields the 3 equations:

$$(u^d+v^d)\, s_{12}^d c_{12}^d (c_{13}^d)^2 = (u^d-v^d)\left[-s_{13}^d\sin 2\theta_{23}^d\cos 2\theta_{12}^d - s_{12}^d c_{12}^d\cos 2\theta_{23}^d(1+(s_{13}^d)^2)\right]; \tag{65a}$$

$$(u^d+v^d)\, c_{12}^d s_{13}^d c_{13}^d = (u^d-v^d)\, c_{13}^d\left[c_{12}^d s_{13}^d\cos 2\theta_{23}^d - s_{12}^d\sin 2\theta_{23}^d\right]; \tag{65b}$$

$$(u^d+v^d)\, s_{12}^d s_{13}^d c_{13}^d = (u^d-v^d)\, c_{13}^d\left[s_{12}^d s_{13}^d\cos 2\theta_{23}^d + c_{12}^d\sin 2\theta_{23}^d\right], \tag{65c}$$

that we now solve.

First make the ratio of (65b) and (65c). For $s_{13}^d c_{13}^d \neq 0$ and $c_{13}^d\neq 0$, it yields $\frac{s_{12}^d}{c_{12}^d} \overset{s_{13}^d c_{13}^d\neq 0, c_{13}^d\neq 0}{=} \frac{c_{12}^d s_{13}^d\cos 2\theta_{23}^d - s_{12}^d\sin 2\theta_{23}^d}{s_{12}^d s_{13}^d\cos 2\theta_{23}^d + c_{12}^d\sin 2\theta_{23}^d} \Rightarrow \sin 2\theta_{23}^d = 0 \Rightarrow \theta_{23}^d = 0 \;or\; \frac{\pi}{2}$.

For $\theta_{23}^d = 0$ (65) become

$$(u^d+v^d)\, s_{12}^d c_{12}^d (c_{13}^d)^2 = -(u^d-v^d)\, s_{12}^d c_{12}^d(1+(s_{13}^d)^2); \tag{66a}$$

$$(u^d+v^d)\, c_{12}^d s_{13}^d c_{13}^d = (u^d-v^d)\, c_{12}^d s_{13}^d c_{13}^d; \tag{66b}$$

$$(u^d+v^d)\, s_{12}^d s_{13}^d c_{13}^d = (u^d-v^d)\, s_{12}^d s_{13}^d c_{13}^d. \tag{66c}$$

Since $s_{13}^d c_{13}^d \neq 0$, (66b) and (66c) demand $v^d = 0$ which, plugged into (66a), yields $2u^d s_{12}^d c_{12}^d = 0$, requiring either $u^d = 0$ or $[s_{12}^d c_{12}^d = 0 \Rightarrow \theta_{12}^d = 0 \;or\; \theta_{12}^d = \frac{\pi}{2}]$.

For $\theta_{23}^d = \frac{\pi}{2}$ (65) become

$$(u^d+v^d)\, s_{12}^d c_{12}^d (c_{13}^d)^2 = (u^d-v^d)\, s_{12}^d c_{12}^d(1+(s_{13}^d)^2); \tag{67a}$$

$$(u^d+v^d)\,c^d_{12}s^d_{13}c^d_{13} = -(u^d-v^d)\,c^d_{12}s^d_{13}c^d_{13}; \tag{67b}$$
$$(u^d+v^d)\,s^d_{12}s^d_{13}c^d_{13} = -(u^d-v^d)\,s^d_{12}s^d_{13}c^d_{13}. \tag{67c}$$

Since $s^d_{13}c^d_{13} \neq 0$, (67b) and (67c) demand $u^d=0$ which, plugged into (67a), yields $2v^d s^d_{12}c^d_{12}=0$, requiring either $v^d=0$ or $[s^d_{12}c_{12}=0 \Rightarrow \theta^d_{12}=0 \; or \; \theta^d_{12}=\frac{\pi}{2}]$.
$c^d_{13}=0$ is a trivial solution of (65b) and (65c); (65a) becomes, then,
$(u^d-v^d)\left[\sin 2\theta^d_{23}\cos 2\theta^d_{12}+\sin 2\theta^d_{12}\cos 2\theta^d_{23}\right]=0 \Rightarrow \theta^d_{12}=-\theta^d_{23}+\frac{n\pi}{2} \; or \; u=v$.
For $s^d_{13}=0$, (65b) and (65c) entail again $[\sin 2\theta^d_{23}=0 \Rightarrow \theta^d_{23}=0 \; or \; \theta^d_{23}=\frac{\pi}{2}]$, or $u^d=v^d$, while (65a) becomes $(u^d+v^d)s^d_{12}c^d_{12}=-(u^d-v^d)s^d_{12}c^d_{12}\cos 2\theta^d_{23}$. For $u^d=v^d$ this requires $\theta^d_{12}=0 \; or \; \frac{\pi}{2}$, for $\theta_{23}=0$, this requires either $u^d=0$ or $[\theta^d_{12}=0 \; or \; \frac{\pi}{2}]$ and, for $\theta^d_{23}=\frac{\pi}{2}$, this requires either $v^d=0$ or $[\theta^d_{12}=0 \; or \; \frac{\pi}{2}]$.
To summarize, the solutions to the suppression of FCNC at 1-loop in bare flavor space are the following:

$$\begin{array}{rcl}
(a) & : & u^d=0=v^d \; (\Leftrightarrow \alpha^d=\beta^d=\gamma^d);\\
(b) & : & \theta^d_{12}=0=\theta^d_{23}=\theta^d_{13} : \text{ general mass-flavor alignment (trivial solution)};\\
(c) & : & \theta^d_{13}=0=\theta^d_{12}, \theta^d_{23}=\dfrac{\pi}{2};\\
(d) & : & \theta^d_{13}=0, \theta^d_{23}=\dfrac{\pi}{2}=\theta^d_{12};\\
(e) & : & \theta^d_{13}=0=\theta^d_{23}, \theta^d_{12}=\dfrac{\pi}{2};\\
(f) & : & \theta^d_{13}=\dfrac{\pi}{2}, \theta^d_{23}=-\theta^d_{12}+\dfrac{n\pi}{2};\\
(g) & : & \theta^d_{13}=\dfrac{\pi}{2}, u^d=v^d \; (\Leftrightarrow \beta^d=\gamma^d);\\
(h) & : & \theta^d_{12}=0=\theta^d_{23}, v^d=0 \; (\Leftrightarrow \alpha^d=\gamma^d);\\
(i) & : & \theta^d_{12}=0, \theta^d_{23}=\dfrac{\pi}{2}, u^d=0 \; (\Leftrightarrow \alpha^d=\beta^d);\\
(j) & : & \theta^d_{12}=0=\theta^d_{13}, u^d=v^d \; (\Leftrightarrow \beta^d=\gamma^d);\\
(k) & : & \theta^d_{12}=\dfrac{\pi}{2}, \theta^d_{23}=0, v^d=0 \; (\Leftrightarrow \alpha^d=\gamma^d);\\
(l) & : & \theta^d_{12}=\dfrac{\pi}{2}=\theta^d_{23}, u^d=0 \; (\Leftrightarrow \alpha^d=\beta^d);\\
(m) & : & \theta^d_{12}=\dfrac{\pi}{2}, \theta^d_{13}=0, u^d=v^d \; (\Leftrightarrow \beta^d=\gamma^d);\\
(n) & : & \theta^d_{23}=0=\theta^d_{13}, u^d=0 \; (\Leftrightarrow \alpha^d=\beta^d);\\
(o) & : & \theta^d_{23}=\dfrac{\pi}{2}, \theta^d_{13}=0, v^d=0 \; (\Leftrightarrow \alpha^d=\gamma^d).
\end{array} \tag{68}$$

Note that $\theta_{13}=0=\theta_{23}$ is a solution of (65) included in (g). These solutions correspond to the following $\mathcal{C}_{d0}$'s:

$$(a) \overset{\alpha^d=\beta^d=\gamma^d}{\rightarrow} any;\; (b)\rightarrow \mathbb{I};\; (c)\rightarrow \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix};\; (d)\rightarrow \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix};\; (e)\rightarrow \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix};$$

$$(f) \overset{n=1}{\rightarrow} \begin{pmatrix} 0 & 0 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix} \overset{n=2}{or} \begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix} \overset{n=3}{or} \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix};\; (g) \overset{\beta^d=\gamma^d}{\rightarrow} \begin{pmatrix} 0 & 0 & 1\\ -s^d_{12+23} & c^d_{12+23} & 0\\ -c^d_{12+23} & -s^d_{12+23} & 0\end{pmatrix};$$

$$(h) \overset{\alpha^d=\gamma^d}{\rightarrow} \mathcal{R}_{13};\; (i) \overset{\alpha^d=\beta^d}{\rightarrow} \begin{pmatrix} c^d_{13} & 0 & s^d_{13}\\ -s^d_{13} & 0 & c^d_{13}\\ 0 & -1 & 0\end{pmatrix};\; (j) \overset{\beta^d=\gamma^d}{\rightarrow} \mathcal{R}_{23};\; (k) \overset{\alpha^d=\gamma^d}{\rightarrow} \begin{pmatrix} 0 & c^d_{13} & s^d_{13}\\ -1 & 0 & 0\\ 0 & -s^d_{13} & c^d_{13}\end{pmatrix};$$

$$(l) \overset{\alpha^d=\beta^d}{\rightarrow} \begin{pmatrix} 0 & c^d_{13} & s^d_{13} \\ 0 & -s^d_{13} & c^d_{13} \\ 1 & 0 & 0 \end{pmatrix};\ (m) \overset{\beta^d=\gamma^d}{\rightarrow} \begin{pmatrix} 0 & 1 & 0 \\ -c^d_{23} & 0 & s^d_{23} \\ s^d_{23} & 0 & c^d_{23} \end{pmatrix};\ (n) \overset{\alpha^d=\beta^d}{\rightarrow} \mathcal{R}_{12};\ (o) \overset{\alpha^d=\gamma^d}{\rightarrow} \begin{pmatrix} c^d_{12} & s^d_{12} & 0 \\ 0 & 0 & 1 \\ s^d_{12} & -c^d_{12} & 0 \end{pmatrix}. \tag{69}$$

Similar formulæ are obtained in the (u, c, t) sector. The relevant parameters will be then given a superscript "u" instead of "d".

We see that the configurations that suppress FCNC are described by two possible sets of conditions: the ones which concern the (d, s, d) mixing angles θ^d_{ij}, fixing the mass-flavor relations in this channel (partial or total alignment *etc*), and the ones concerning $\alpha^d, \beta^d, \gamma^d$ which establish connections between the masses (fermions, W, μ) and the CKM angles θ_{ij}, δ. Solution (a) is of the second type; (b), (c), (d), (e), (f) are of the first type; all others are mixed.

The physical mixing patterns that are observed exhibit, in addition to approximate alignment as one goes up the generations, some peculiar values of some of CKM angles. This is why we shall focus in the following on the solutions that possibly constrain the latter, *i.e.* (a) and (g) to (o).

The conditions of the second type may not be possible to achieve. The first task is accordingly to scrutinize the conditions $\alpha = \beta, \beta = \gamma, \alpha = \gamma$ in both channels, (d, s, b) and (u, c, t), and to select the ones that can be fulfilled. If, for example, in the (d, s, b) channel, only $\alpha^d = \beta^d$ can be achieved, one has to choose among the 7 solutions (b), (c), (d), (e), (f), (i), (l), (n). The first four are very constrained solutions. For (b), there is total mass-flavour alignment in this sector. For (c), (d) and (e), the 3 angles in the (d, s, b) sector are either vanishing of equal to $\frac{\pi}{2}$. For (f), $\theta^d_{13} = \frac{\pi}{2}$ while the sum of the 2 other angles is a multiple of $\frac{\pi}{2}$. In (i) and (l), θ^d_{12} and θ^d_{23} are constrained, respectively to 0 or $\frac{\pi}{2}$ and to $\frac{\pi}{2}$, leaving θ^d_{13} free, while in (n), θ^d_{13} and θ^d_{23} are both constrained to 0, while θ^d_{12} is left free.

Still with the example of the (d, s, b) channel, the conditions $\alpha^d = \beta^d, \beta^d = \gamma^d, \alpha^d = \gamma^d$ write respectively

$$\begin{aligned} \mathcal{A}^{\pm}_{dd} + \mathcal{A}^3_{dd} &= \mathcal{A}^{\pm}_{ss} + \mathcal{A}^3_{ss}, \\ \mathcal{A}^{\pm}_{ss} + \mathcal{A}^3_{ss} &= \mathcal{A}^{\pm}_{bb} + \mathcal{A}^3_{bb}, \\ \mathcal{A}^{\pm}_{dd} + \mathcal{A}^3_{dd} &= \mathcal{A}^{\pm}_{bb} + \mathcal{A}^3_{bb}, \end{aligned} \tag{70}$$

in which, like in subsection 2.2, $\mathcal{A}^{\pm}_{ii}$ and $\mathcal{A}^3_{ii}$ denote the 1-loop amplitudes for the diagonal transition $i \to i$ mediated respectively by $W^{\pm}$ and W^3.

It is simple matter, using the unitarity of V, to get

$$\mathcal{A}^3_{ii} - \mathcal{A}^3_{jj} = \frac{1}{2}(h_i - h_j). \tag{71}$$

$$\begin{aligned} \mathcal{A}^{\pm}_{dd} &= |V_{ud}|^2(h_u - h_t) + |V_{cd}|^2(h_c - h_t), \\ \mathcal{A}^{\pm}_{ss} &= |V_{us}|^2(h_u - h_t) + |V_{cs}|^2(h_c - h_t), \\ \mathcal{A}^{\pm}_{bb} &= |V_{ub}|^2(h_u - h_t) + |V_{cb}|^2(h_c - h_t), \\ \mathcal{A}^{\pm}_{uu} &= |V_{ud}|^2(h_d - h_b) + |V_{us}|^2(h_s - h_b), \\ \mathcal{A}^{\pm}_{cc} &= |V_{cd}|^2(h_d - h_b) + |V_{cs}|^2(h_s - h_b), \\ \mathcal{A}^{\pm}_{tt} &= |V_{td}|^2(h_d - h_b) + |V_{ts}|^2(h_s - h_b). \end{aligned} \tag{72}$$

The 6 non-trivial conditions (3 in the (d, s, b) sector and 3 in the (u, c, t) sector) that we need consider write accordingly

$$\alpha^u = \beta^u : \frac{1}{2}(h_d - h_s) + (|V_{ud}|^2 - |V_{us}|^2)(h_u - h_t) + (|V_{cd}|^2 - |V_{cs}|^2)(h_c - h_t) = 0, \tag{73a}$$

$$\beta^u = \gamma^u : \frac{1}{2}(h_s - h_b) + (|V_{us}|^2 - |V_{ub}|^2)(h_u - h_t) + (|V_{cs}|^2 - |V_{cb}|^2)(h_c - h_t) = 0, \tag{73b}$$

$$\alpha^u = \gamma^u : \frac{1}{2}(h_d - h_b) + (|V_{ud}|^2 - |V_{ub}|^2)(h_u - h_t) + (|V_{cd}|^2 - |V_{cb}|^2)(h_c - h_t) = 0, \tag{73c}$$

$$\alpha^d = \beta^d : \frac{1}{2}(h_u - h_c) + (|V_{ud}|^2 - |V_{cd}|^2)(h_d - h_b) + (|V_{us}|^2 - |V_{cs}|^2)(h_s - h_b) = 0, \tag{73d}$$

$$\beta^d = \gamma^d : \frac{1}{2}(h_c - h_t) + (|V_{cd}|^2 - |V_{td}|^2)(h_d - h_b) + (|V_{cs}|^2 - |V_{ts}|^2)(h_s - h_b) = 0, \tag{73e}$$

$$\alpha^d = \gamma^d : \frac{1}{2}(h_u - h_t) + (|V_{ud}|^2 - |V_{td}|^2)(h_d - h_b) + (|V_{us}|^2 - |V_{ts}|^2)(h_s - h_b) = 0. \tag{73f}$$

The 6 equations (73) include only 2 pairs of independent conditions ((73a) +(73a)=(73c), (73d)+(73e)=(73f)). The particular case of 2 generations, that we studied before, is easily recovered. One has, then, $|V_{ud}|^2 = c_c^2 = |V_{cs}|^2, |V_{us}|^2 = s_c^2 = |V_{cd}|^2$. (73) shrinks to

$$\begin{aligned} \alpha^u = \beta^u : \frac{1}{2}(h_d - h_s) + (c_c^2 - s_c^2)(h_u - h_c) &= 0, \\ \alpha^d = \beta^d : \frac{1}{2}(h_u - h_c) + (c_c^2 - s_c^2)(h_d - h_s) &= 0, \end{aligned} \tag{74}$$

of which only the first can be realized, leading to a large (quasi-maximal) Cabibbo angle, and leaving mass-flavor alignment as the only possibility in the (u, c) sector.

7.2 Coping with the top quark: analytic expressions for $(h_i - h_t)$

The approximate expression of $(h_i - h_j)$ for $m_i^2, m_j^2, p^2 \ll m_W^2$ is given by (13). It is valid for u, d, s, c, b quarks, all leptons, but it is not valid when the top quark is involved. In this case, an approximate expression for $(h_i - h_t)$ can still be obtained from (12), which is valid for for $m_i^2, p^2 \ll m_W^2$, and keeps exact in the top quark mass dependence m_t:

$$\begin{aligned} h_i - h_t &\approx \frac{g^2}{4}\frac{i}{16\pi^2}\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_i^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right), \\ t_{terms} &\approx \frac{2}{3}\frac{m_t^2}{m_W^2} + \frac{7}{2}\frac{m_W^2}{m_W^2 - m_t^2} + \frac{1}{4}\left(5 - \frac{m_t^2}{m_W^2}\right)\frac{m_t^2}{m_W^2 - m_t^2} \\ &\quad + 2\frac{m_W^2 \ln\frac{m_W^2}{\mu^2} - m_t^2 \ln\frac{m_t^2}{\mu^2}}{m_W^2 - m_t^2} - \frac{1}{2}\left(2 + \frac{m_t^2}{m_W^2}\right)\frac{m_W^4 \ln\frac{m_W^2}{\mu^2} - m_t^4 \ln\frac{m_t^2}{\mu^2}}{(m_W^2 - m_t^2)^2} \\ &\quad - \frac{m_t^2}{m_W^2}\frac{1}{(m_W^2 - m_t^2)^2}\left(-\frac{11 m_W^4 - 7 m_W^2 m_t^2 + 2 m_t^4}{6} + \frac{m_W^6 \ln\frac{m_W^2}{\mu^2} + (-3 m_t^2 m_W^4 + 3 m_t^4 m_W^2 - m_t^6)\ln\frac{m_t^2}{\mu^2}}{m_W^2 - m_t^2}\right). \end{aligned} \tag{75}$$

When m_t becomes larger and larger, t_{terms} scale like

$$t_{terms} \overset{m_t \gg m_W}{\sim} \frac{m_t^2}{m_W^2}\left(\frac{7}{12} - \frac{1}{2}\ln\frac{m_t^2}{\mu^2}\right). \tag{76}$$

In practice, according to (73), one needs $(h_u - h_t)$ and $(h_c - h_t)$.

7.3 Solving the constraints for 3 generations of quarks

The CKM matrix we parametrize as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\, e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\, e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{77}$$

such that

$$\begin{aligned}
|V_{ud}|^2 - |V_{us}|^2 &= c_{13}^2 \cos 2\theta_{12};\\
|V_{cd}|^2 - |V_{cs}|^2 &= \cos 2\theta_{12}(-c_{23}^2 + s_{13}^2 s_{23}^2) + \sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{us}|^2 - |V_{ub}|^2 &= s_{12}^2 c_{13}^2 - s_{13}^2;\\
|V_{ud}|^2 - |V_{ub}|^2 &= c_{12}^2 c_{13}^2 - s_{13}^2;\\
|V_{cs}|^2 - |V_{cb}|^2 &= c_{12}^2 c_{23}^2 + s_{23}^2(-c_{13}^2 + s_{12}^2 s_{13}^2) - \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{ud}|^2 - |V_{cd}|^2 &= c_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - s_{12}^2 c_{23}^2 - \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{us}|^2 - |V_{cs}|^2 &= s_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - c_{12}^2 c_{23}^2 + \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{cd}|^2 - |V_{cb}|^2 &= s_{12}^2 c_{23}^2 + s_{23}^2(c_{12}^2 s_{13}^2 - c_{13}^2) + \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{cd}|^2 - |V_{td}|^2 &= \cos 2\theta_{23}(s_{12}^2 - c_{12}^2 s_{13}^2) + \sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{cs}|^2 - |V_{ts}|^2 &= \cos 2\theta_{23}(c_{12}^2 - s_{12}^2 s_{13}^2) - \sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{ud}|^2 - |V_{td}|^2 &= c_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - s_{12}^2 s_{23}^2 + \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta;\\
|V_{us}|^2 - |V_{ts}|^2 &= s_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - c_{12}^2 s_{23}^2 - \frac{1}{2}\sin 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta.
\end{aligned} \tag{78}$$

The constraints (73) become (we remind that t_{terms} is given in (75))

$$\begin{aligned}
\alpha^u = \beta^u \quad : \quad & \frac{1}{2}\frac{m_d^2 - m_s^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) = \\
& -c_{13}^2 \cos 2\theta_{12}\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_u^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right] \\
& - \left[\cos 2\theta_{12}(-c_{23}^2 + s_{13}^2 s_{23}^2) + \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right] \\
& \qquad \left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_c^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right];
\end{aligned} \tag{79a}$$

$$\begin{aligned}
\beta^u = \gamma^u \quad : \quad & \frac{1}{2}\frac{m_s^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) = \\
& -(s_{12}^2 c_{13}^2 - s_{13}^2)\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_u^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right] \\
& - \left[c_{12}^2 c_{23}^2 + s_{23}^2(-c_{13}^2 + s_{12}^2 s_{13}^2) - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right] \\
& \qquad \left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_c^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right];
\end{aligned} \tag{79b}$$

$$\alpha^u = \gamma^u \quad : \quad \frac{1}{2}\frac{m_d^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) =$$

$$
-(c_{12}^2 c_{13}^2 - s_{13}^2)\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_u^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right]
$$

$$
-\left[s_{12}^2 c_{23}^2 + s_{23}^2(c_{12}^2 s_{13}^2 - c_{13}^2) + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]
$$

$$
\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_c^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right]; \tag{79c}
$$

$$
\alpha^d = \beta^d \quad : \quad \frac{1}{2}(m_u^2 - m_c^2) = -(m_d^2 - m_s^2)\left[c_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - s_{12}^2 c_{23}^2 - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]
$$

$$
-(m_s^2 - m_b^2)\left[s_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - c_{12}^2 c_{23}^2 + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]; \tag{79d}
$$

$$
\beta^d = \gamma^d \quad : \quad \frac{1}{2}\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_c^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right] =
$$

$$
-\frac{m_d^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)\left[\cos 2\theta_{23}(s_{12}^2 - c_{12}^2 s_{13}^2) + \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]
$$

$$
-\frac{m_s^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)\left[\cos 2\theta_{23}(c_{12}^2 - s_{12}^2 s_{13}^2) - \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]; \tag{79e}
$$

$$
\alpha^d = \gamma^d \quad : \quad \frac{1}{2}\left[\left(-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_u^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) + t_{terms}\right)\right] =
$$

$$
-\frac{m_d^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)\left[c_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - s_{12}^2 s_{23}^2 + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]
$$

$$
-\frac{m_s^2 - m_b^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)\left[s_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - c_{12}^2 s_{23}^2 - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right]. \tag{79f}
$$

Notice that (79d) is the only equation which is not influenced by the large mass of the top quark.

For $\theta_{23} = 0 = \theta_{13}$, (79a) reduces to $\frac{1}{2}(m_d^2 - m_s^2) = (m_c^2 - m_u^2)\cos 2\theta_{12}$, which is the constraint on the Cabibbo angle when 2 generations only are present (the first of eqs. (74)).

Once the masses of the fermions, the one of the W gauge boson, and the renormalization scale μ are fixed, they constitute a system of 4 equations for the 4 CKM angles $\theta_{12}, \theta_{23}, \theta_{13}$ and δ.

Some simplifications can be performed. First, even in the large interval $\mu \in [100\, MeV, m_W]$, the t_{terms} largely dominate over $\frac{m_{u,c}^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)$, at least by a factor 1000. The latter can thus always be neglected. The same t_{terms} dominate over $\ln\frac{m_W^2}{\mu^2}$ by at least a factor 3, and over $\frac{3}{2}$ by at least a factor 6. It is accordingly a reasonable approximation to only consider their contribution inside the corresponding [ ] brackets. Secondly, it is also reasonable to neglect $m_d^2 \ll m_b^2, m_s^2 \ll m_b^2, m_u^2 \ll m_c^2$ and, even, $m_d^2 \ll m_s^2$. The system (79) then simplifies to

$$
\alpha^u = \beta^u \quad : \quad \frac{m_s^2}{m_W^2}\left(-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) \approx 2\, t_{terms}\left((c_{13}^2 - c_{23}^2 + s_{23}^2 s_{13}^2)\cos 2\theta_{12} + s_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta\right); \tag{80a}
$$

$$\beta^u = \gamma^u \quad : \quad \frac{m_b^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) \approx 2\,t_{terms}\Big(s_{12}^2 c_{13}^2 - s_{13}^2 + c_{12}^2 c_{23}^2 + s_{23}^2(-c_{13}^2 + s_{12}^2 s_{13}^2) \\ -\frac{1}{2} s_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta\Big); \tag{80b}$$

$$\alpha^u = \gamma^u \quad : \quad \frac{m_b^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) \approx 2\,t_{terms}\Big(c_{12}^2 c_{13}^2 - s_{13}^2 + s_{12}^2 c_{23}^2 + s_{23}^2(c_{12}^2 s_{13}^2 - c_{13}^2) \\ +\frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\Big); \tag{80c}$$

$$\alpha^d = \beta^d \quad : \quad m_c^2 \approx -2\Bigg(m_s^2\Big[\cos 2\theta_{12}\left(c_{23}^2 + c_{13}^2 - s_{23}^2 s_{13}^2\right) - s_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta\Big] \\ +m_b^2\Big[s_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - c_{12}^2 c_{23}^2 + \frac{1}{2}\, s_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta\Big]\Bigg); \tag{80d}$$

$$\beta^d = \gamma^d \quad : \quad t_{terms} \approx 2\frac{m_b^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) c_{13}^2\cos 2\theta_{23}; \tag{80e}$$

$$\alpha^d = \gamma^d \quad : \quad t_{terms} \approx 2\frac{m_b^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right)\left(c_{13}^2 - s_{13}^2 - s_{23}^2 c_{13}^2\right). \tag{80f}$$

It is important to stress that the system (80) is only approximate, while (79) is exact; this why, in particular, while the simultaneous fulfillment of (79b) and (79c) (resp. (79e) and (79f)) entails that of (79a) (resp. (79d)), the same does not occur for (80b), (80c) and (80a) (resp. (80e), (80f) and (80d)) .

As a short numerical calculation shows, (80e) can never be satisfied, because it would correspond to $|c_{13}^2\cos 2\theta_{23}| > 300$ (still for $\mu \in [100\,MeV, m_W]$). The same argumentation shows that (80f) cannot be satisfied either. So, in the (d,s,b) sector, only $\alpha^d = \beta^d$ can eventually be satisfied and solutions (b), (c), (d), (e), (f), (i), (l), (n) are the only ones that should be considered.

Summing (80b) and (80c) yields a constraint which does not include θ_{12} nor δ:

$$\frac{1}{t_{terms}}\frac{m_b^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) = 3c_{13}^2(1+s_{23}^2) - 1, \tag{81}$$

such that the quantity $3c_{13}^2(1+s_{23}^2)-1$ must be a small number, the modulus of which does not exceed $1.5\,10^{-3}$. The condition $0 \leq s_{23}^2 \leq 1$ entails

$$\frac{1}{6} \leq c_{13}^2 \leq \frac{1}{3} \overset{\theta_{13}\in[0,\frac{\pi}{2}]}{\Rightarrow} 55^0 \leq \theta_{13} \leq 66^o. \tag{82}$$

which is not compatible with the observed value of θ_{13} in the CKM matrix. (80b) and (80c) are not either individually compatible with the observed values of the CKM angles. Indeed, plugging in these values, their r.h.s. come close to $2t_{terms}$, which is much larger than their l.h.s.

Let us now consider (80a) and (80d). Since, for $\mu \in [100\,MeV, m_W]$, $\frac{m_s^2}{m_W^2}\left(-\frac{17}{4}+\frac{3}{2}\ln\frac{m_W^2}{\mu^2}\right) \ll 2t_{terms}$, (80a) rewrites

$$\alpha^u = \beta^u : (s_{23}^2 - s_{13}^2 + s_{23}^2 s_{13}^2)\cos 2\theta_{12} + s_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta \approx 0, \tag{83}$$

which is presumably only trustable for $\delta = 0$ since we did not introduce any CP-violating phase in the partial rotations $\mathcal{R}_{12}, \mathcal{R}_{23}, \mathcal{R}_{13}$.

In case (80a) and (80d) are simultaneously satisfied, eliminating the CP-violating phase δ between the two of them yields

$$m_c^2 \approx -2\left(2\,m_s^2 c_{13}^2 \cos 2\theta_{12} + m_b^2\Big[s_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - c_{12}^2 c_{23}^2 - \frac{1}{2}\,(c_{13}^2 - c_{23}^2 + s_{23}^2 s_{13}^2)\cos 2\theta_{12}\Big]\right), \tag{84}$$

from which one deduces that very small values of θ_{23} and θ_{13}, like observed in the quark sector, are only compatible with θ_{12} quasi-maximal: $\cos 2\theta_{12} \approx \frac{m_c^2}{2(m_b^2 - 2m_s^2)}$ ($\theta_{12} \approx 44^o$), which is not the observed value ($\theta_{12} \approx 13^o$) of the Cabibbo angle. Consequently, a rather small Cabibbo angle can only be achieved if at least one among the two angles θ_{23} and θ_{13} is not very small. As we saw by summing (80b) and (80c), this must be the case of θ_{13}. From (81) and (84), one gets, after neglecting $\frac{2m_s^2}{3(1+s_{23}^2)} \ll \frac{m_b^2}{3}$

$$s_{23}^2 \approx \frac{4}{3}c_{12}^2 + \frac{m_c^2}{2m_b^2} \approx \frac{4}{3}c_{12}^2 + 4.5\,10^{-2}, \tag{85}$$

which entails in particular $s_{23}^2 \geq 4.5\,10^{-2} \Rightarrow \theta_{23} \geq 12^o$ and $c_{12}^2 \leq \frac{3}{4} \Rightarrow \theta_{12} \geq 30^o$.

To summarize, the only equations that can eventually be simultaneously satisfied are (73a) to (73d). They lead to CKM angles which are not the ones observed in the quark sector, and which are all fairly large (except θ_{23} which can go as low as 12^o).

There are of course other possibilities, which are to be looked for among the solutions (a) to (o) in each of the two sectors (d,s,b) and (u,c,t).

It is appropriate to consider solution (b) which means global mass-flavor alignment, in one of the two sectors, first, for example (u,c,t). The only left over constraint from the demanded suppression of extra FCNC is accordingly (80d), which corresponds to $\alpha^d = \beta^d$ (we recall that (80e) and (80f) can never be satisfied). Only solutions (b), (c), (d), (e), (f), (i), (l), (n) are thus to be considered. They apply to mixing angles of the (d,s,b) sector, but these can be identified with CKM angle due to the alignment in the u-type sector. (b) corresponds to global mass-flavor alignment in the (d,s,b) sector, too. (c), (d), (e), (f) correspond to the CKM matrices represented in (69). They offer no special interest, mixing angles being 0 or $\frac{\pi}{2}$. (i), with $\theta_{12} = 0, \theta_{23} = \frac{\pi}{2}$, yields $\cos 2\theta_{13} \approx -\frac{m_c^2}{2m_s^2}$ which is impossible because it is > 1. (l), with $\theta_{12} = \frac{\pi}{2} = \theta_{23}$, corresponds to $\cos 2\theta_{23} = -\frac{m_c^2}{2(m_b^2 - m_s^2)}$ very small, such that θ_{13} is close to maximal. (n), with $\theta_{13} = 0 = \theta_{23}$, corresponds to $\cos 2\theta_{12} \approx -\frac{m_c^2}{2(m_b^2 - 2m_s^2)}$, such that θ_{12} is close to maximal.

Let us then choose global mass-flavor alignment in the (d,s,b) sector. Only (73a), (73b) and (73c) can then be considered as eventual constraints to suppress extra FCNC, and we shall consider them for $\delta = 0$, neglecting CP-violation effects. If the 3 of them are realized, we have already seen that θ_{13} will be large $55^o \leq \theta_{13} \leq 66^o$. Since this is in contradiction with observation, we have to relax at least one of the three constraints. Since they are not independent, at least 2 of them must be relaxed, otherwise the 3rd would be automatically satisfied. Keeping only (73b) or only (73c) cannot accommodate for very small θ_{13} and θ_{23} (see (68)), such that, if one looks for solutions close to reality, it looks appropriate to relax both of them and only keep (73a), associated with the constraint $\alpha^u = \beta^u$. Among the solutions associated with the latter, (n) (see (68)) is specially worth investigating because the exact suppression of extra FCNC corresponds then to vanishing θ_{23} and θ_{13}. In this case, as we already mentioned, (79a) reduces to the 2-generation constraint $\cos 2\theta_{12} = \frac{1}{2}\frac{m_d^2 - m_s^2}{m_c^2 - m_u^2}$, which corresponds to a Cabibbo angle close to maximal. A not fully complete suppression can be thought to possibly accommodate for small values of θ_{23} and θ_{13}.

Instead of working on the approximate system (80), let us rather consider the exact one (79) and, more specifically, (79a) in a realistic situation when θ_{23} and θ_{13} are not strictly vanishing but only very small. Solution (n) is not, then, exactly satisfied at 1-loop, but it could be at higher orders. More precisely, let us determine which values of θ_{12} are compatible with (79a) and realistic values of θ_{23} and θ_{13}. (79a) rewrites (for $\delta = 0$)

$$\frac{1}{2}\frac{m_d^2 - m_s^2}{m_W^2} - \frac{m_c^2 - m_u^2}{m_W^2}\cos 2\theta_{12} \approx$$
$$-\frac{m_c^2}{m_W^2}\sin 2\theta_{23}s_{13}\sin 2\theta_{12} + \left(s_{13}^2\frac{m_u^2}{m_W^2} - s_{23}^2(1+s_{13}^2)\frac{m_c^2}{m_W^2}\right)\cos 2\theta_{12}$$
$$+\Big((s_{13}^2 - s_{23}^2 - s_{13}^2 s_{23}^2)\cos 2\theta_{12} - \sin 2\theta_{23}s_{13}\sin 2\theta_{12}\Big)\, T(m_t, m_W, \mu),$$

$$T(m_t, m_W, \mu) = \frac{-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + t_{terms}}{-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}} \overset{m_t \gg m_W}{\sim} \frac{-\frac{3}{2} - \ln\frac{m_W^2}{\mu^2} + \frac{m_t^2}{m_W^2}\left(\frac{7}{12} - \frac{1}{2}\ln\frac{m_t^2}{\mu^2}\right)}{-\frac{17}{4} + \frac{3}{2}\ln\frac{m_W^2}{\mu^2}}. \tag{86}$$

The expression for t_{terms} is given in (75) and its behaviour as m_t grows, which we used in the r.h.s. of (86), has been given in (76).

The prediction for 2 generations is obtained by putting the r.h.s. of (86) to 0, that is, for example, by setting $s_{23} = 0 = s_{13}$.

The modulus of T is larger than 1.45 as soon as $\mu \geq 10\, MeV$, while $\frac{m_c^2}{m_W^2} \approx 3.5\, 10^{-4}$. So, we can neglect $2\frac{m_c^2}{m_W^2}s_{23}s_{13}\sin 2\theta_{12}$ with respect to $2Ts_{23}s_{13}\sin 2\theta_{12}$ is the r.h.s. of (86). As for the terms proportional to $\cos 2\theta_{12}$, $s_{23}^2 s_{13}^2\frac{m_c^2}{m_W^2} \ll s_{23}^2 s_{13}^2 T$, such that (86) can be approximated by

$$\frac{1}{2}\frac{m_d^2 - m_s^2}{m_W^2} - \frac{m_c^2 - m_u^2}{m_W^2}\cos 2\theta_{12} \approx \frac{s_{13}^2 m_u^2 - s_{23}^2 m_c^2}{m_W^2}\cos 2\theta_{12}$$
$$+\Big((s_{13}^2 - s_{23}^2 - s_{13}^2 s_{23}^2)\cos 2\theta_{12} - \sin 2\theta_{23}s_{13}\sin 2\theta_{12}\Big)T(m_t, m_W, \mu). \tag{87}$$

The vanishing of the l.h.s. of (87) is the condition for no extra FCNC for 2 generations only (see (74)). Its modulus is always smaller than $\frac{m_c^2}{m_W^2}$. So is the modulus of the first term in the r.h.s. of (87). At the opposite, the modulus of T is, as we mentioned, larger than 1.45 for $\mu \geq 10\, MeV$. Accordingly, the coefficient of T in (87) should be very small, which writes

$$\left|(s_{13}^2 - s_{23}^2 - s_{13}^2 s_{23}^2)\cos 2\theta_{12} - \sin 2\theta_{23}s_{13}\sin 2\theta_{12}\right| \approx \left|\frac{\frac{1}{2}\frac{m_d^2 - m_s^2}{m_W^2} - \frac{m_c^2(1+s_{23}^2) - m_u^2(1+s_{13}^2)}{m_W^2}\cos 2\theta_{12}}{T(m_t, m_W, \mu)}\right| \leq 2\, 10^{-4} \ll 1. \tag{88}$$

There are two ways to consider the relation (88):

* the first is to directly plug in the experimental values for s_{23} and s_{13} and see whether they correspond to a suitable value of the Cabibbo angle θ_{12}. Experimentally, $\theta_{12} \approx 13^o$, $s_{13} \approx V_{ub} \approx 4.1\, 10^{-3}$, $s_{23} \approx V_{cb} \approx 42\, 10^{-3}$, such that the l.h.s. of (88) is found approximately equal to $1.5\, 10^{-3}$ instead of a few 10^{-4}. The agreement is far from being good;

* eqs. (76) and (86) show that the l.h.s. of (88) scales, when m_t gets larger and larger, like $\lambda_1\dfrac{m_c^2}{m_t^2\left(1 + \lambda_2 \ln\frac{m_t^2}{\mu^2}\right)}$, and goes accordingly to 0 when the hierarchy $\frac{m_t}{m_c}$ increases. When m_t gets very large $m_t \gg m_W$, the CKM angles must therefore satisfy the condition

$$\tan 2\theta_{12} \approx \frac{s_{13}^2 - s_{23}^2 - s_{13}^2 s_{23}^2}{s_{13}\sin 2\theta_{23}}. \tag{89}$$

If one plugs in (89) the observed values of θ_{12} and θ_{13}, one finds that this corresponds to $\theta_{12} \approx 38^o$. Reciprocally, plugging in a realistic value $|\tan 2\theta_{12}| \approx \frac{1}{2}$ for the Cabibbo angle, one gets $s_{13} \approx$

$\frac{\sqrt{5}-1}{2}\tan\theta_{23} \approx .618\tan\theta_{23}$. Though the precise values disagree with experiment, they satisfy, as observed, $\theta_{13} < \theta_{23}$.

As we show now, a very heavy top quark tends to drag the value of the Cabibbo angle down from quasi-maximal (which is the prediction for 2 generations) to a smaller value. For that purpose, let us perform the same study assuming now that $m_t \ll m_W$, only, for example, slightly heavier than the bottom quark. Instead of the system (79), eqs. (73) now yield

$$\begin{aligned}\alpha^u = \beta^u \quad : \quad & \frac{1}{2}(m_d^2 - m_s^2) = -c_{13}^2 \cos 2\theta_{12}(m_u^2 - m_t^2) \\ & - \left[\cos 2\theta_{12}(-c_{23}^2 + s_{13}^2 s_{23}^2) + \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_c^2 - m_t^2);\end{aligned} \tag{90a}$$

$$\begin{aligned}\beta^u = \gamma^u \quad : \quad & \frac{1}{2}(m_s^2 - m_b^2) = -(s_{12}^2 c_{13}^2 - s_{13}^2)(m_u^2 - m_t^2) \\ & - \left[c_{12}^2 c_{23}^2 + s_{23}^2(-c_{13}^2 + s_{12}^2 s_{13}^2) - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_c^2 - m_t^2);\end{aligned} \tag{90b}$$

$$\begin{aligned}\alpha^u = \gamma^u \quad : \quad & \frac{1}{2}(m_d^2 - m_b^2) = -(c_{12}^2 c_{13}^2 - s_{13}^2)(m_u^2 - m_t^2) \\ & - \left[s_{12}^2 c_{23}^2 + s_{23}^2(c_{12}^2 s_{13}^2 - c_{13}^2) + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_c^2 - m_t^2);\end{aligned} \tag{90c}$$

$$\begin{aligned}\alpha^d = \beta^d \quad : \quad & \frac{1}{2}(m_u^2 - m_c^2) = -\left[c_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - s_{12}^2 c_{23}^2 - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_d^2 - m_s^2) \\ & - \left[s_{12}^2(c_{13}^2 - s_{23}^2 s_{13}^2) - c_{12}^2 c_{23}^2 + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_s^2 - m_b^2);\end{aligned} \tag{90d}$$

$$\begin{aligned}\beta^d = \gamma^d \quad : \quad & \frac{1}{2}(m_c^2 - m_t^2) = -\left[\cos 2\theta_{23}(s_{12}^2 - c_{12}^2 s_{13}^2) + \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_d^2 - m_b^2) \\ & - \left[\cos 2\theta_{23}(c_{12}^2 - s_{12}^2 s_{13}^2) - \sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_s^2 - m_b^2);\end{aligned} \tag{90e}$$

$$\begin{aligned}\alpha^d = \gamma^d \quad : \quad & \frac{1}{2}(m_u^2 - m_t^2) = -\left[c_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - s_{12}^2 s_{23}^2 + \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_d^2 - m_b^2) \\ & - \left[s_{12}^2(c_{13}^2 - c_{23}^2 s_{13}^2) - c_{12}^2 s_{23}^2 - \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23} s_{13}\cos\delta\right](m_s^2 - m_b^2).\end{aligned} \tag{90f}$$

Neglecting $m_d \ll m_s$, $m_s \ll m_b$, $m_d \ll m_b$, $m_u \ll m_c$, $m_u \ll m_t$ and supposing also that $m_c \ll m_t$, (90a) approximates to

$$\frac{1}{2}(m_d^2 - m_s^2) - (m_c^2 - m_u^2)\cos 2\theta_{12} \approx m_t^2\left[(-s_{13}^2 c_{23}^2 + s_{23}^2)\cos 2\theta_{12} + s_{13}\sin 2\theta_{23}\sin 2\theta_{12}\right]; \tag{91}$$

the sum of (90b) and (90c) yields

$$\frac{m_b^2}{m_t^2} \approx 1 - 3\,c_{13}^2 c_{23}^2; \tag{92}$$

Eq. (90e) becomes

$$\frac{m_b^2}{m_t^2} \approx -\frac{1}{2}\,\frac{1}{c_{13}^2\cos 2\theta_{23}}, \tag{93}$$

and (90f)

$$\frac{m_b^2}{m_t^2} \approx -\frac{1}{2}\,\frac{1}{c_{13}^2 c_{23}^2 - s_{13}^2} \equiv -\frac{1}{2}\,\frac{1}{\cos 2\theta_{13} - s_{23}^2 c_{13}^2}. \tag{94}$$

Eqs. (93) and (94) can only be simultaneously verified if $c_{13}^2 \approx 1$, such that $\theta_{13} \approx 0$. Plugging this result into (92) requires $c_{23}^2 \approx \frac{1}{3}\left(1 - \frac{m_b^2}{m_t^2}\right)$. This entails $\theta_{23} \geq \arccos\frac{1}{\sqrt{3}} \approx 54^o$. Then, (91) yields $\frac{1}{2}(m_d^2 - m_s^2) = \cos 2\theta_{12}\left[(m_c^2 - m_u^2) + m_t^2\left(\frac{2}{3} + \frac{m_b^2}{3m_t^2}\right)\right]$. Because of the term proportional to m_t^2, the corresponding modulus of $\cos 2\theta_{12}$ gets accordingly smaller than for 2 generations; this corresponds to a larger Cabibbo angle, thus still closer to maximal. This is the opposite of what happens when the top quark gets much heavier than the W. So, as announced, by going across the electroweak scale and getting more and more massive, the top quark shifts down the modulus of the 1-loop Cabibbo angle with respect to the 2-generation case.

7.4 Solving the constraints for 3 generations of leptons

The case that we just investigated, when all fermion masses for 3 generations stand below the W scale corresponds *a priori* to the leptonic sector. There, while one knows that $m_e \ll m_\mu \ll m_\tau$, our knowledge about the neutrino messes essentially concerns the extreme smallness of their differences [14][7].

This is why all 3 equations (90a), (90b) and (90c), in which the differences of neutrino mass squared occurring in the r.h.s.'s are always much smaller than the ones of charged leptons occurring in the l.h.s.'s, can never be satisfied. This leaves only (b), (c), (d), (e) and (f) as possible solutions of (68) for charged leptons. (b) corresponds to general mass-flavor alignment; in (c) and (e), 1 flavor state is aligned with the corresponding mass state, while exact swapping, 2 by 2, occurs for the remaining 4 states; for example,, for (c), $e_f = e_m, \mu_f = \tau_m, \tau_f = -\mu_m$; in (d) and (f), the 6 states are swapped 2 by 2, with no alignment for any pair. This corroborates the common, but never demonstrated statement, that charged leptons do not oscillate [13].

As for equations (90d), (90e) and (90f), the extreme smallness of their l.h.s.'s forces their r.h.s.'s to be practically vanishing. (90e) and (90f) become respectively

$$m_\tau^2 c_{13}^2 \cos 2\theta_{23} \approx 0 \tag{95}$$

and

$$m_\tau^2(\cos 2\theta_{13} - s_{23}^2 c_{13}^2) = 0. \tag{96}$$

Excluding $\theta_{13} = \pm\frac{\pi}{2}$, (95) yields $\cos 2\theta_{23} = 0 \Rightarrow \theta_{23}$ maximal $\Rightarrow c_{23}^2 = \frac{1}{2} = s_{23}^2$; when plugged into (96), this entails $\tan^2\theta_{13} = c_{23}^2 = \frac{1}{2} \Rightarrow \theta_{13} \approx \pm 35^o$. One has $s_{13} \approx \pm.577, c_{13} \approx .816$. When the numerical values of s_{23}^2 and c_{23}^2 are plugged in (90d), it becomes

$$\begin{aligned}&\left[c_{12}^2\left(c_{13}^2 - \frac{1}{2}s_{13}^2\right) - \frac{1}{2}s_{12}^2 - \frac{1}{2}\sin 2\theta_{12} s_{13}\cos\delta\right](m_e^2 - m_\mu^2)\\ &+\left[s_{12}^2\left(c_{13}^2 - \frac{1}{2}c_{13}^2\right) + \frac{1}{2}s_{12}^2 - \frac{1}{2}\sin 2\theta_{12} s_{13}\cos\delta\right](m_\mu^2 - m_\tau^2) = 0.\end{aligned} \tag{97}$$

Neglecting $m_e \ll m_\mu$, $m_\mu \ll m_\tau$, the approximate solution of (97) writes $\tan\theta_{12} \approx -\frac{2s_{13}\cos\delta}{3c_{13}^2} \overset{\delta=0}{\approx} \mp.577 \Rightarrow |\theta_{12}| \approx 30^o$.

The values that we have found for θ_{12} and θ_{23} are very close to the experimental values. We furthermore predict $|\theta_{13}| \approx 35^o$, which is still to be measured in future experiments.

Before concluding on the neutrino sector, and in relation with the common prejudice that θ_{13} is small, let us check that no other solution among (68) can accommodate for such a small angle. The only one that could eventually fit is (o). Then, the equivalent of (90f) writes (taking $\theta_{23} = \frac{\pi}{2}, \theta_{13} \approx 0$)

$$\frac{1}{2}(m_e^2 - m_\tau^2) \approx (c_{12}^2 - s_{12}^2)(m_{\nu_e}^2 - m_{\nu_\mu}^2), \tag{98}$$

which, due to the strong hierarchy $(m_\tau^2 - m_e^2) \gg (m_{\nu_\mu}^2 - m_{\nu_e}^2)$, has no solution.

8 Outlook

We have paid in this study special attention to 1-loop transitions and to their role in fermionic mixing. They spoil the diagonality of kinetic terms which must be, first, cast back into their canonical form before the mass matrix is re-diagonalized and orthogonal mass eigenstates suitably determined.

A first property that we encountered is that, for non-degenerate systems, bare mass states and 1-loop mass states are non-unitarily related.

A second property is that the 1-loop mixing matrix $\mathfrak{C}(p^2)$ occurring in charged currents (Cabibbo, PMNS $\ldots$) stays unitary at $\mathcal{O}(g^2)$.

The third point concerns the 1-loop value of the CKM angles, and their equivalent for leptons. The classical standard model does not provide any hint that could help connecting masses and mixing angles. Therefore, most investigations have concerned special structures or textures of classical mass matrices that could eventually be explained by subtle and broken symmetries, the origin of which being itself lying presumably “beyond the standard model” [15]. To make it short, there are more free parameters than masses and mixing angles in the classical standard model, and one is looking for constraints that reduce their number, so as to, ultimately, put masses and mixing in one-to-one correspondence.

The classical SM is like a smooth polished sphere and it is extremely hard to find a defect or asperity to break in and put it in jeopardy. The diagonalization of classical mass matrix by bi-unitary transformations is perfectly adequate and kinetic terms keep unchanged since they are chosen from the beginning to be proportional to the unit matrix. Through the covariant derivative, this form of the kinetic terms dictates that of gauge currents, in particular neutral currents, for which FCNC can only occur at 1-loop with the so-called “Cabibbo suppression”, “unfortunately” very successful, too. The last cornerstone which bears this elegant construction is the unitarity of the Cabibbo (CKM) matrix, which ensures, in bare mass space, the closure of the $SU(2)_L$ algebra, when embedded in $SU(2n_f)$ (n_f is the number of flavors), on a diagonal T^3 generator, in which both $n_f \times n_f$ sub-blocks are proportional to the unit matrix. The grain of salt that may grip this beautiful machinery is, for example, if kinetic terms are no longer diagonal. Through gauge invariance and the covariant derivative, neutral gauge currents are then no longer diagonal either: extra FCNC have been generated, which we know is extremely dangerous because these are very constrained by experiments. Now, experiments concern physical states, which are defined at the poles of the full propagator. Since for them the standard CKM phenomenology is perfectly successful, we think rather unlikely that “something goes wrong” in this space. Getting, there, a suitable $SU(2)_L$ algebra which closes on “good old diagonal T^3” is therefore a suitable goal to achieve. This goes, for example, with a unitary renormalized CKM matrix. Then, where can things go “wrong”? If not in physical mass space, maybe in bare mass or flavor space, the two of them being unitary related. Classically, physical and bare mass spaces are identical. But they are not at 1-loop. Extra FCNC can be generated in bare mass space if they are no longer unitarily related with physical states. Since physical states are constructed to be orthogonal (one diagonalizes the renormalized quadratic Lagrangian), a non-unitary relation with bare mass states can only occur if the latter are non-orthogonal *i.e.* if there exists non-diagonal transitions among them. This is the point that we exploited in this work. Bare mass or flavor states are no longer orthogonal at 1-loop, and they can never be, because of mass splittings. We show that it is much better, for the stability of corrections, to introduce counterterms “à la Shabalin”, but they cannot completely restore the orthogonality of bare mass states on mass shell, because the different mass-shells do not coincide. So, some trace of non-orthogonality always subsists in this space, and thus, a slight non-unitarity in the connection between physical states and bare mass (or flavor) states always remains, too. Therefore, in these last bases, some extra FCNC are always generated at 1-loop with respect to the classical SM. This means in particular that, in there, the gauge structure (generators, closure on nice $T^3 \ldots$) is not perturbatively stable. It might be possible to cope with this, but, in this work, we chose to be very conservative and to perturbatively preserve the structure of the Lagrangian that was chosen at the classical level. We therefore asked that these extra FCNC vanish or, at least, be strongly damped. Since they depend on the classical CKM (or PMNS) angles, on the fermion and W masses (and on one renormalization scheme μ), the constraints that we obtained connect these parameters.

Shabalin's counterterms play a decisive role. They are very seldom introduced, though they were already proved to be determinant in the calculation of the electric dipole moment of the quarks [6]. We have shown that, in their absence, quantum corrections to mixing angle go all the more out of control as fermions come closer to degeneracy. One then faces technical problems such that results of perturbative calculations cannot manifestly be trusted. As we explicitly saw in the case of two generations, they furthermore allow for non-trivial solutions to the suppression of extra FCNC. In their absence, while mass-flavor quasi-alignment occurs for the fermion pair the farthest from degeneracy, no special condition arises concerning the Cabibbo angle. Instead, in their presence, in addition to the trivial, aligned, solution, quasi-maximal mixing for the fermion pair the closest to degeneracy, associated with mass-flavor quasi-alignment for the other pair comes out as another suitable possibility. In the case of three generations, we systematically introduced them, which had also the technical advantage to largely ease the calculations because they "nearly" cancel non-diagonal kinetic terms.

The results that we obtained in the leptonic sector have the twofold advantage to be quite encouraging (nice agreement for θ_{12} and θ_{23}) and also easily falsifiable in coming neutrinos experiments since we also predict a large $\theta_{13} \approx 35^o$. The quark sector looks more problematic. We have been unable to get a small Cabibbo angle, and the other two CKM angles also come out much too large. The only encouraging point is the role of a heavy quark $m_t \ll m_W$ which decreases the value of the 1-loop θ_{12} possibly down to 38^o. Unfortunately. this value is still much too large. So, what is happening in the hadronic sector [13]? The role of leptons and quarks seem to have been interchanged because, while, previously, the large values of the neutrino mixing angles were problematic, it is now the small values of the ones of quarks that are hard to accounted for. One could be tempted to invoke the eventual existence of more super-heavy fermions that could eventually drag down still more the renormalized mixing angles. But the complexity of calculations in the presence of extra generations of fermions rises so dramatically that it can only be the object of a (long and tedious) forthcoming work. More simply, the small measured values could just be thought of as second order corrections to the trivial solution with general mass-flavor alignment for all quark species. Unfortunately, 2-loops calculations in the presence of Shabalin's like counterterms stand at present also beyond our technical abilities.

Should physics "beyond the standard model" be invoked? Suppose that the leptonic θ_{13} is measured to be large $\approx 35^o$ as we predict. The conservative conjecture of ours that Shabalin's counterterms are enough to cancel extra FCNC with respect to the standard CKM phenomenology looks then reliable and presumably carries some part of truth. Then, if BSM physics is needed, it is to find a theoretical more sound basis to this statement. The situation looks different for hadrons, but one should not be too much in a hurry to invoke BSM physics before calculations of 2-loop corrections have been achieved.

We end up this work by pointing out at some differences with previous approaches of the subject. This study is based on the mandatory (re)-diagonalization of the sum of kinetic and mass terms to suitably determine an orthogonal set of mass eigenstates. While this requirement is always and simply taken care of at the classical level by a bi-unitary diagonalization of the mass matrix, it is generally overlooked as soon as radiative corrections are concerned [16] [17] [18] [19] [20]. In particular, only considering self-mass contributions to determine the renormalized mass states from the renormalized mass matrix exposes to the problem that they are not orthogonal since there still exist kinetic-like transitions between them. We show that the re-diagonalization of kinetic terms can have important effects.
* First, and this is not a new result [8] [9] [10] [11] but we confirm it, bare mass (or flavor) states are non-unitarily related to 1-loop mass eigenstates for non-degenerate systems. It turns out however, that, unlike individual mixing matrices, the 1-loop Cabibbo matrix $\mathfrak{C}(p^2)$ occurring in charged currents stays unitary (see however the caveat in appendix A.1). It is a consequence of gauge invariance, which in particular connects, through the covariant derivative of fermion fields, kinetic terms to gauge currents, both at the classical level and including radiative corrections. The expression of the 1-loop Cabibbo matrix $\mathfrak{C}(p^2)$

[13] A solution has been proposed in [16] in which, in the quark sector, (d, s) and (u, c) mixing angles largely cancel each other while, in the lepton sector, the opposite occurs.

is thus directly dictated by that of the 1-loop kinetic terms, which is one more reason to pay a special attention to them;
* then, by a cascade of mechanisms, mixing angles close to maximal naturally appear if one wants to preserve the standard CKM phenomenology.

We hope to have convinced the reader that a reasonable argumentation exists that can account for large mixing angles by linking them with small mass splittings without invoking BSM physics from the start. If explaining both leptonic and hadronic sectors still remains a challenge, at least 2 among the 3 neutrino mixing angles come out with magnitudes which are close to their measured values. Future lies accordingly in the hands and both experimentalists and theorists, the first, in p[[articular, to measure the leptonic θ_{13}, and the second to estimate higher order corrections to mass-flavor quasi-alignment of quarks and see whether their can account for the smallness of the CKM angles.

Acknowledgments: It is a pleasure to thank M.I. Vysotsky for comments and advice.

A The dependence on p^2. Canceling transitions between non-degenerate physical states

A.1 Non-orthogonality of non-degenerate physical states

Eqs. (10), (11), (12), (13), which we obtained in the absence of Shabalin's counterterms, are only valid when $p^2 \ll m_W^2$, but it must kept in mind that all formuæ depend on p^2, even though this dependence becomes very weak when $p^2 \ll m_W^2$.

At the price, when no counterterms are introduced, of a high instability in the vicinity of degeneracy (see subsection 5.1) the Cabibbo procedure can be rescued and a p^2-dependent, unitary renormalized Cabibbo matrix $\mathfrak{C}(p^2, \ldots)$ be defined. The 1-loop effective Lagrangian is made diagonal (see section 2) in the basis $d_{mL}(p^2, \ldots), s_{mL}(p^2, \ldots)$, in which p_μ stands for the common 4-momentum of d and s (see Fig. 2). This means that there exist no more non-diagonal transitions between them, such that $d_{mL}(p^2, \ldots)$ and $s_{mL}(p^2, \ldots)$ are, by definition, orthogonal at 1-loop. However, as soon as a mass splitting exists, both cannot be simultaneously on mass-shell and the physical fermions

$$\begin{aligned}
d_{mL}^{phys} &\equiv d_{mL}\big(p^2 = \mu_d^2(p^2)\big) = [(\mathcal{V}_d\,\mathcal{R}(\xi_d))^{-1}]_{11}\big(p^2 = \mu_d^2(p^2)\big)\ d_{mL}^0 + [(\mathcal{V}_d\,\mathcal{R}(\xi_d))^{-1}]_{12}\big(p^2 = \mu_d^2(p^2)\big)\ s_{mL}^0,\\
s_{mL}^{phys} &\equiv s_{mL}\big(p^2 = \mu_s^2(p^2)\big) = [(\mathcal{V}_d\,\mathcal{R}(\xi_d))^{-1}]_{21}\big(p^2 = \mu_s^2(p^2)\big)\ d_{mL}^0 + [(\mathcal{V}_d\,\mathcal{R}(\xi_d))^{-1}]_{22}\big(p^2 = \mu_s^2(p^2)\big)\ s_{mL}^0,
\end{aligned} \tag{99}$$

which belong to two different sets of orthogonal states, are themselves expected to be non-orthogonal. So, unless subtle cancellations take place, non-diagonal transitions are expected to occur among them, which is akin to saying that the 1-loop Lagrangian, despite it has been built by diagonalization, is itself not diagonal when re-expressed in terms physical non-degenerate eigenstates. At the same time, unlike $\mathfrak{C}(p^2)$ in (38), which is defined for an overall global p^2, the "on mass-shell" Cabibbo matrix is expected to exhibit some slight non-unitarity [9] [10] [11].

More specifically, the 1-loop quadratic effective Lagrangian (kinetic and mass terms) can be generically rewritten in the basis of physical eigenstates

$$
\begin{aligned}
\mathcal{L}^{1-loop} \;=\;& \left(\begin{array}{cc}\overline{d_{mL}^{phys}} & \overline{s_{mL}^{phys}}\end{array}\right)\not p \left(\begin{array}{cc} g_1(p^2) & g_2(p^2)\\ g_3(p^2) & g_4(p^2)\end{array}\right)\left(\begin{array}{c} d_{mL}^{phys}\\ s_{mL}^{phys}\end{array}\right) + \left(\begin{array}{cc}\overline{d_{mR}^{phys}} & \overline{s_{mR}^{phys}}\end{array}\right)\not p\, \mathbb{I} \left(\begin{array}{c} d_{mR}^{phys}\\ s_{mR}^{phys}\end{array}\right)\\
&- \left(\begin{array}{cc}\overline{d_{mL}^{phys}} & \overline{s_{mL}^{phys}}\end{array}\right)\left(\begin{array}{cc} \rho_1(p^2) & \rho_2(p^2)\\ \rho_3(p^2) & \rho_4(p^2)\end{array}\right)\left(\begin{array}{c} d_{mR}^{phys}\\ s_{mR}^{phys}\end{array}\right)\\
&- \left(\begin{array}{cc}\overline{d_{mR}^{phys}} & \overline{s_{mR}^{phys}}\end{array}\right)\left(\begin{array}{cc} \sigma_1(p^2) & \sigma_2(p^2)\\ \sigma_3(p^2) & \sigma_4(p^2)\end{array}\right)\left(\begin{array}{c} d_{mL}^{phys}\\ s_{mL}^{phys}\end{array}\right) + \ldots
\end{aligned}
\qquad (100)
$$

Indeed, combined with (23) which relates bare mass states to 1-loop mass eigenstates, (99) entails that the coefficients of the linear relation between the latter and physical states are functions of $(p^2, \ldots)$. Hermiticity requires the (supposedly real and presumably $\mathcal{O}(g^2)$) quantities $g_2, g_3, \sigma_2, \sigma_3, \rho_2, \rho_3$ to satisfy the relations $g_3 = g_2, \rho_2 = \sigma_3, \rho_3 = \sigma_2$. Furthermore, since right-handed fermions are not concerned by 1-loop transitions, $(1+\gamma^5)d_m^{phys} = (1+\gamma^5)d_m^0$ and $(1+\gamma^5)s_m^{phys} = (1+\gamma^5)s_m^0$.

A.2 Recovering orthogonality on mass-shell

Whether Shabalin's counterterms are included or not, the same technique of diagonalizing the effective, p^2-dependent, quadratic Lagrangian yields by definition orthogonal 1-loop mass eigenstates $d_m(p^2), s_m(p^2)$, which are however not the physical states. Therefore, an argumentation similar to the one used, in the absence of counterterms, in subsection A.1, can be invoked in their presence: non-diagonal transitions between physical mass eigenstates at 1-loop are expected to occur, and, when expressed in terms of them, the effective Lagrangian at 1-loop is expected to also be of the form (100).

When classical physical states (which are nothing more than bare mass states) and 1-loop physical states do not drastically differ (for example would they differ by perturbative amounts), one expects the non-diagonal "scalar products" not to be drastically different either within the two sets. This cannot be guaranteed in the absence of Shabalin's counterterms because of the non-perturbative nature of the link that occurs, then, between the two sets. In their presence, instead, they only differ by "small amounts" and the above property is expected to be true: since non-diagonal transitions between bare mass states are, then, canceled at $\mathcal{O}(g^2)$, this is certainly also true among 1-loop physical states.

Higher order non-diagonal transitions that still exist, in the presence of Shabalin's counterterms, between on mass-shell 1-loop $s_{mL}(p^2)$ and $d_{mL}(p^2)$ can always be canceled by another set of counterterms. This is shown in subsection A.3 below. However, being presumably of order higher than g^2, they should only be introduced in the framework of a 2-loop calculation, which is out of the scope of the present work.

A.3 Expression of the additional counterterms in the basis of physical states

From any Lagrangian of the form (100), on-diagonal, p^2-dependent transitions between on mass-shell fermions, like $\mu \leftrightarrow e$ are expected This can be embarrassing since defining on mass-shell muon and electron as asymptotic states seems then problematic. They can however be themselves canceled by introducing counterterms, as follows. But for the fact that we are now working in the space of physical states, the procedure is formally similar to the one used in [6] to determine Shabalin's counterterms, which we recalled in section 5.2 (see also [10], appendix A). Canceling, for example, (on mass-shell s) $\to$ (on mass-shell d) transitions can be done by adding to (100) four kinetic and mass-like counterterms, concerning both chiralities of fermions:

$$
-\mathcal{A}_d\, \overline{d_m^{phys}}\not p(1-\gamma^5)s_m^{phys} - \mathcal{B}_d\, \overline{d_m^{phys}}(1-\gamma^5)s_m^{phys} - \mathcal{E}_d\, \overline{d_m^{phys}}\not p(1+\gamma^5)s_m^{phys} - \mathcal{D}_d\, \overline{d_m^{phys}}(1+\gamma^5)s_m^{phys}. \qquad (101)
$$

Since s_m^{phys} is on mass-shell, one gets the condition (we call respectively μ_s and μ_d the 1-loop physical masses of s and d, that is, the square roots of the values of p^2 solutions of $p^2 = \mu_s^2(p^2)$ and $p^2 = \mu_d^2(p^2)$ (see subsection 2.4.2))

$$\begin{aligned} & g_2(\mu_s^2)\,\overline{d_m^{phys}}(1+\gamma^5)\mu_s\, s_m^{phys} - \rho_2(\mu_s^2)\,\overline{d_m^{phys}}(1+\gamma^5)s_m^{phys} - \sigma_2(\mu_s^2)\,\overline{d_m^{phys}}(1-\gamma^5)s_m^{phys} \\ &= \mathcal{A}_d\,\overline{d_m^{phys}}(1+\gamma^5)\mu_s s_m^{phys} + \mathcal{B}_d\,\overline{d_m^{phys}}(1-\gamma^5)s_m^{phys} + \mathcal{E}_d\,\overline{d_m^{phys}}(1-\gamma^5)\mu_s s_m^{phys} + \mathcal{D}_d\,\overline{d_m^{phys}}(1+\gamma^5)s_m^{phys}, \end{aligned} \tag{102}$$

and since d_m^{phys} is also on mass-shell,

$$\begin{aligned} & g_2(\mu_d^2)\,\overline{d_m^{phys}}(1-\gamma^5)\mu_d\, s_m^{phys} - \rho_2(\mu_d^2)\,\overline{d_m^{phys}}(1+\gamma^5)s_m^{phys} - \sigma_2(\mu_d^2)\,\overline{d_m^{phys}}(1-\gamma^5)s_m^{phys} \\ &= \mathcal{A}_d\,\overline{d_m^{phys}}(1-\gamma^5)\mu_d s_m^{phys} + \mathcal{B}_d\,\overline{d_m^{phys}}(1-\gamma^5)s_m^{phys} + \mathcal{E}_d\,\overline{d_m^{phys}}(1+\gamma^5)\mu_d s_m^{phys} + \mathcal{D}_d\,\overline{d_m^{phys}}(1+\gamma^5)s_m^{phys}. \end{aligned} \tag{103}$$

Equating the terms with identical chiralities in (102) and (103) yields the four equations

$$\begin{aligned} \mu_s\, g_2(\mu_s^2) - \rho_2(\mu_s^2) &= \mu_s\mathcal{A}_d + \mathcal{D}_d, \\ -\sigma_2(\mu_s^2) &= \mu_s\mathcal{E}_d + \mathcal{B}_d, \\ \mu_d\, g_2(\mu_d^2) - \sigma_2(\mu_d^2) &= \mu_d\mathcal{A}_d + \mathcal{B}_d, \\ -\rho_2(\mu_d^2) &= \mu_d\mathcal{E}_d + \mathcal{D}_d, \end{aligned} \tag{104}$$

which have the $\mathcal{O}(g^2)$ solutions

$$\begin{aligned} \mathcal{A}_d &= \frac{\mu_s^2\, g_2(\mu_s^2) - \mu_d^2\, g_2(\mu_d^2) + \mu_s\left(\rho_2(\mu_d^2) - \rho_2(\mu_s^2)\right) - \mu_d\left(\sigma_2(\mu_s^2) - \sigma_2(\mu_d^2)\right)}{\mu_s^2 - \mu_d^2}, \\ \mathcal{E}_d &= \frac{\mu_d\mu_s\left(g_2(\mu_s^2) - g_2(\mu_d^2)\right) + \mu_d\left(\rho_2(\mu_d^2) - \rho_2(\mu_s^2)\right) - \mu_s\left(\sigma_2(\mu_s^2) - \sigma_2(\mu_d^2)\right)}{\mu_s^2 - \mu_d^2}, \\ \mathcal{B}_d &= -\sigma_2(\mu_s^2) - \mu_s\mathcal{E}_d, \\ \mathcal{D}_d &= -\rho_2(\mu_d^2) - \mu_d\mathcal{E}_d. \end{aligned} \tag{105}$$

Likewise, four counterterms $\tilde{\mathcal{A}}_d, \tilde{E}_d, \tilde{\mathcal{B}}_d, \tilde{D}_d$ can get rid of the on mass-shell $d_m^{phys} \to s_m^{phys}$ transitions. Hermiticity (see above) constrains them to satisfy $\tilde{\mathcal{A}}_d = \mathcal{A}_d, \tilde{\mathcal{E}}_d = \mathcal{E}_d, \tilde{\mathcal{B}}_d = \mathcal{D}_d, \tilde{\mathcal{D}}_d = \mathcal{B}_d$. Similar additions can be done in the (u, c) sector.

As emphasized at the end of subsection A.2, when Shabalin's counterterms are already present, the additional counterterms invoked here are presumably of higher order in g.

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